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# A Convergent Reformulation of perturbative QCD

Ricardo João Gaio Alves<sup>1</sup>

Centre for Particle Theory

University of Durham

October 2, 2000

A thesis submitted to the University of Durham for the degree of  
Doctor of Philosophy

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<sup>1</sup>The author was the recipient of a scholarship from Programa Praxis XXI-JNICT-Portugal.



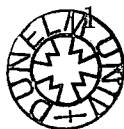
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# Abstract

We present and explore a new formulation of perturbative QCD based not on the renormalised coupling but on the dimensional transmutation parameter of the theory and the property of asymptotic scaling. The approach yields a continued function, the iterated function being that involved in the solution of the two-loop  $\beta$ -function equation. In the so-called large- $b$  limit the continued function reduces to a continued fraction and the successive approximants are diagonal Padé approximants. We investigate numerically the convergence of successive approximants using the leading- $b$  approximation, motivated by renormalons, to model the all-orders result. We consider the Adler  $D$ -function of vacuum polarisation, the Polarised Bjorken and Gross-Llewellyn Smith sum rules, the (unpolarised) Bjorken sum rule, and the Minkowskian quantities  $R_\tau$  and the  $R$ -ratio of  $e^+e^-$  annihilation. In contrast to diagonal Padé approximants the truncated continued function method gives remarkably stable large-order approximants in cases where infra-red renormalon effects are important. We also use the new approach to determine the QCD fundamental parameters from the  $R_\tau$  and the  $R$ -ratio measurements, where we find  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 516 \pm 48$  MeV (which yields  $\alpha_s(\mu = m_\tau) = 0.360_{-0.020}^{+0.021}$ ), and  $\tilde{\Lambda}_{\overline{MS}}^{(5)} = 299_{-7}^{+6}$  MeV (which yields  $\alpha_s(\mu = m_{Z^0}) = 0.1218 \pm 0.0004$ ), respectively. The evolution of the former value to the  $m_{Z^0}$  energy results in  $\alpha_s(\mu = m_{Z^0}) = 0.123 \pm 0.002$ . These values are in line with other determinations available in the literature.

We implement the Complete Renormalisation Group Improvement (CORGI) scheme throughout all the calculations.

We report on how the mathematical concept of Stieltjes series can be used to assess the convergence of Padé approximants of perturbative series. We find that the combinations of  $UV$  renormalons which occur in perturbative QCD may or may not be Stieltjes series depending on the renormalisation scheme used.



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*I acknowledge the receipt of a money grant from the Physics department to cater for supervisory meetings (at CERN) during the second semester of the first year.*

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## Declaration

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

The research reported in this thesis has been carried out in collaboration with Dr. C.J. Maxwell. The publication of the material contained in Chapters 5 and 6 is in preparation.

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# Preface

Perturbative Field Theory has proven to be a useful approach to describe subatomic phenomena. Its theoretical predictions for many experimentally measurable quantities were found to be remarkably good, specially in the domains of QED and the Electroweak model. QCD, the field theory of strong interactions, has provided a less reassuring agreement between theory and experiment. This is generally attributed to the large value of the QCD coupling, and to the lack of a satisfactory theory in the low energy limit, but it has also brought to attention the importance of some conceptual issues that do not cause any difficulties for other field theories. One of these is the assumption, which impregnates all (perturbative) field theories, that perturbative series are convergent. This false premise must be replaced by the more cautious expectation that perturbative series are asymptotically convergent. Another issue is the fact that, contrary to what happens in QED, there is no macroscopic renormalization point to fixate the value of the running coupling. Thus, we are left with an awkward dependence of the QCD results in arbitrary, unphysical parameters. This Thesis addresses both these issues.

The first Chapter of this Thesis intends to give a general overview of QCD. The QCD Lagrangian and its symmetries are described, and the Feynman rules are quoted. The discussion of the issues of regularisation and the Renormalisation Group is extensive, and includes the exact solution of the first two truncations of the RG  $\beta$ -function equation. Anticipating one of the problems this thesis will be concerned with, renormalisation scheme dependence is discussed at length.

Chapter 2 is a self-contained description of both the problems and some of the possible solutions facing large-orders in perturbation theory. It starts by reviewing some of the reasons why field theory series expansions are expected to diverge. The

concept of convergent series is then superseded by the more convenient, and actually more generic one of asymptotic series. Some of the possible ways of making sense out of asymptotic series are afterwards discussed, namely the Borel sum and the Padé approximants. The Chapter ends with a discussion of Stieltjes series, which are shown to be a specially important particular case of asymptotic series. The reasons why QCD observables are not expected to be represented by Stieltjes series are outlined.

Chapter 3 discusses renormalons, which are expected to dominate the large orders of QCD perturbation theory. Two different paths are followed to show how renormalons arise for one specific physical observable. The leading- $b$  approximation, of crucial importance in the field of renormalons, is then examined. The Chapter continues with an exploration of the connections between renormalons and power corrections, and ends with a critical comparison of QED and QCD perturbative series.

Chapter 4 contains the description of a new method to deal with large orders in perturbation theory, suggested by C.J. Maxwell in [1] and based on a continued function construction. Its derivation is carried out. How the CORGI renormalisation scheme [2] can be used in the leading- $b$  approximation is outlined. The method is then tested in the simplest most favorable case of an alternating-sign factorial series, where one finds that, despite the absence of an absolute proof of convergence, numerical evidence suggests that the continued function method gives convergent results. The main purpose of this thesis shall be to apply this method to several QCD problems.

Chapter 5 contains the application of our resummation method to the Adler function, to the polarised Bjorken sum rule, and to the unpolarised Bjorken sum rule. It starts with the formal explanation of how our resummation method is applied to the problems under consideration. Comparisons are made with Padé approximants and the Borel sum as resummation methods, and with the term-by-term sum of the perturbation series.

Chapter 6 finally compares the performance of the method under study with experimental data. This is done for both the  $\tau$  lepton width decay to hadrons, and the R-ratio  $e^+e^-$  annihilation into hadrons. This paves the way for a determination

of the QCD fundamental parameter(s).

In Chapter 7 we review our results and outline our conclusions.

# Chapter 1

## Elements of QCD

The aim of this Chapter is to describe the main aspects of the theory of *Quantum Chromodynamics* (QCD). The QCD Lagrangian is introduced, the corresponding Feynman rules are stated, and a taste for the problems that lie beyond leading order calculations is provided by a brief discussion of radiative corrections to the gluon propagator at one loop order. This is followed by a discussion of the solutions to the lowest order truncations of the renormalisation group  $\beta$ -equation, and by an extensive discussion of the renormalisation scheme dependence problem.

The first three sections rely partly on references [3, 4], and benefited also from [5, 6] for the most theoretical aspects of field theory.

### 1.1 The QCD Lagrangian

We know from phenomenological evidence that the fundamental particles of QCD, the *quarks*, have an internal quantum number which can take three different values (usually designated as *colours*,  $N$ ). This implies the existence of at most nine colour-carriers, the *gluons*, one of which, the colour singlet, has not been observed in nature. Therefore, the correct symmetry group for the strong interactions must be  $SU(3)$ . The quarks shall be represented by vectors in the three-dimensional space of colour, and since the quarks are fermions, each of these vectors will be a Dirac spinor  $\Psi(x)$  in the Lorentz space. The  $SU(3)$  vectors transform under an irreducible representation of  $SU(3)$ :

$$\Psi \longrightarrow \Psi' = U(x)\Psi, \quad (1.1)$$

where the  $U$  are  $3 \times 3$  unitary matrices ( $UU^\dagger = 1$ ), with  $|U|=1$ , which can be written as

$$U(x) = e^{iT^a \theta^a(x)}, \quad (1.2)$$

the  $T^a$  being  $SU(3)$  generators (usually given in terms of the Gell-Mann matrices  $\lambda^a$ ,  $T^a = \frac{1}{2}\lambda^a$ ). These generators obey the commutation relations

$$[T^a, T^b] = if^{abc}T^c, \quad (1.3)$$

where the  $SU(3)$  structure constants  $f^{abc}$  are real and antisymmetric. So, it can be said that the  $N^2 - 1$  gluons  $G_\mu(x)$  ( $\equiv G_\mu^a(x)T^a$ ) are in the *adjoint* representation of  $SU(3)$ , whereas the quarks are in the *fundamental* representation. It is conventionally chosen that

$$Tr(T^a T^b) = T_f \delta^{ab} \quad \left(T_f = \frac{1}{2}\right). \quad (1.4)$$

With this choice, one has the following important relations:

$$T_{ij}^a T_{jk}^a = C_F \delta_{ik} \quad \left(C_F = \frac{N^2 - 1}{2N}\right), \quad (1.5)$$

$$f^{abc} f^{abd} = C_A \delta^{cd} \quad (C_A = N). \quad (1.6)$$

### 1.1.1 The QCD free Lagrangian

The free Lagrangian density describes the propagation of the quarks (massive spin  $\frac{1}{2}$  particles), and the gluons (massless spin 1 particles):

$$\mathcal{L} = \sum_f \bar{\Psi}_f^a (i\not{\partial} - m_f)^{ab} \Psi_f^b - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}. \quad (1.7)$$

The sum in the first term is over  $N_f$  quark flavours, and the notation  $\not{B} = B_\mu \gamma^\mu$  is used, where  $\gamma^\mu$  are the Dirac matrices, which satisfy the anticommutation relations  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  (the metric of the space-time is  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ ). As it

stands, this Lagrangian density does not describe the interactions between quarks and gluons, but it already contains the gluonic self-interactions, which arise from the field strength tensor  $F_{\mu\nu}^a$ :

$$F_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c, \quad (1.8)$$

where  $g_s$  is the interaction coupling strength. The mentioned interactions can be read from the terms which are trilinear and quartic in the fields in the gluonic part of (1.7):

$$\begin{aligned} F_{\mu\nu}^a F^{a,\mu\nu} &= (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a)(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) \\ &+ 2g_s f^{abc}(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a)G^{b,\mu}G^{c,\nu} + g_s^2 f^{abc}f^{ade}G_\mu^b G_\nu^c G^{d,\mu}G^{e,\nu}, \end{aligned} \quad (1.9)$$

whereas the terms in the first line above describe the propagation of gluons.

### 1.1.2 Local gauge invariance

If we require the free Lagrangian (1.7) to be invariant under the local gauge group transformation (1.1), and the corresponding transformation on the Dirac conjugate  $\bar{\Psi} (\equiv \Psi^\dagger \gamma^0)$  is also to hold, one finds that the fermionic term in the Lagrangian (1.7) is not invariant unless we define a *covariant derivative*:

$$D_\mu \equiv \partial_\mu - ig_s G_\mu^a T^a, \quad (1.10)$$

with  $G_\mu$  transforming as

$$G_\mu \longrightarrow G'_\mu = G_\mu - \frac{i}{g_s} U(\partial_\mu U^{-1}). \quad (1.11)$$

The gluonic term in the Lagrangian (1.7), being a trace, was already of an invariant form under the  $SU(3)$  group gauge transformations. Gauge invariance also ensures that the gluons are massless, since a mass term for the gluons, necessarily of the form

$$m^2 G_\mu G^\mu, \quad (1.12)$$

would not be gauge invariant.

### 1.1.3 Gauge-fixing and ghost Lagrangian

A gluon propagator calculated from the first line in (1.9) would have no inverse. This can easily be seen if we notice that the momentum space operator proportional to the gluon propagator would be  $g^{\mu\nu}k^2 - k^\mu k^\nu$ , a projection operator which acts on the transverse states alone. To have an invertible operator, we would need an extra term proportional to  $k^\mu k^\nu$ , which is the form of a projection operator acting on the longitudinal states<sup>1</sup>. These considerations suggest that the gauge can be fixed if one adds to the Lagrangian a new term of the form

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\zeta}(\partial^\mu G_\mu^a)(\partial^\nu G_\nu^a). \quad (1.13)$$

The gluon propagator has now an additional  $\zeta k^\mu k^\nu$  factor, and it is thus invertible. However, physical quantities calculated from the Lagrangian should be independent of the unphysical parameter  $\zeta$ . Thus, its choice is arbitrary. The choices  $\zeta = 0$  and  $\zeta = 1$  are known, respectively, as the *Landau-Lorentz gauge* and the *'t Hooft-Feynman gauge*. At this point, it would be most easily seen using the path-integral formalism that one also has to introduce additional fields, usually known as *ghost fields*:

$$\mathcal{L}_{\text{ghosts}} = -\partial_\mu \bar{\eta}^a \mathcal{D}^\mu \eta^a \quad (\mathcal{D}_\mu \eta^a \equiv \partial_\mu \eta^a - g_s f^{abc} \eta^b G_\mu^c), \quad (1.14)$$

where the fictitious fields  $\eta^a, \bar{\eta}^a$  are massless, scalar fields, thus spinless, but anti-commuting and therefore obeying a Fermi statistics. As it is clear from inspection of  $\mathcal{L}_{\text{ghosts}}$ , these fields only occur in loops in the gluon propagator. This concludes the choice of gauge.

The gauges described above are known as *covariant gauges*. For certain purposes, it is sometimes useful to choose the *axial* class of gauges, where there are no ghost fields, and one imposes  $\eta^\mu G_\mu^a = 0$ ,  $\eta^\mu$  being a fixed four-vector. The Lagrangian is altered by a term

---

<sup>1</sup>In the *Hamiltonian* formulation of Quantum Chromodynamics field theory, this problem expresses itself in the existence of *four* polarisation states for the gluon, whereas an off-shell boson must have only two.



$$\mathcal{L}_{gauge-fixing} = -\frac{1}{2\zeta}(\eta^\mu G_\mu^a)(\eta^\nu G_\nu^a), \quad (1.15)$$

Choosing the gauge now amounts to specifying the vector  $\eta$ . If  $\eta^2 = 0$ , we have a *light-like gauge*. If  $\eta^2 \neq 0$ , we have a *planar gauge*. Needless to say, being added by a  $\zeta\eta^\mu\eta^\nu$  term, the operator to which the gluon propagator is proportional again becomes invertible.

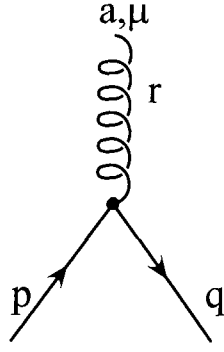
## 1.2 The Feynman rules

The considerations of the previous section lead to a full Lagrangian which, on a generic covariant gauge, is explicitly

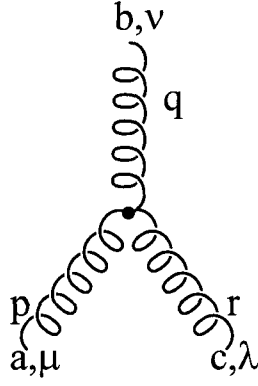
$$\mathcal{L} = \sum_f \bar{\Psi}_f^a (i\not{D} - m_f)^{ab} \Psi_f^b - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} - \frac{1}{2\zeta} (\partial^\mu G_\mu^a)(\partial^\nu G_\nu^a) - \partial_\mu \bar{\eta}^a \mathcal{D}^\mu \eta^a. \quad (1.16)$$

The Feynman rules necessary to perform the calculations of QCD perturbation theory can be deduced from the (chosen dimensionless) action  $S = -i \int d^4x \mathcal{L}(x)$  using the path-integral formalism. This will not be done here. We shall simply quote the Feynman rules.

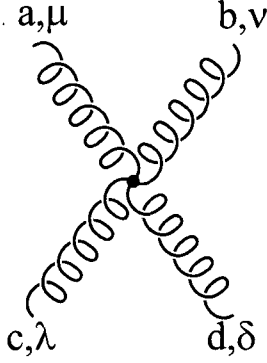
The Feynman rules provide us with a recipe to do perturbative QCD calculations. For each fermionic or bosonic line in a given Feynman diagram, one replaces the corresponding propagator as specified in Figure 1.2. As already mentioned before, the propagators are derived from the terms which are bilinear in the fields. The other terms (trilinear and quartic in the fields) give us the interactions, with the physical vertices being replaced by the corresponding expressions from Figure 1.1. These are the quark-gluon-antiquark ( $qg\bar{q}$ ) vertex, and the self-interactions of the gluons, which can be trilinear ( $ggg$ ) or quartic ( $gggg$ ). The vertex between a gluon and two ghosts on the top of Figure 1.2 stems from the fact that we have chosen to work on a covariant gauge. Ghosts only occur as loops in the self-energy of the gluon. Since they were introduced as Grassman variables, these loops need a minus sign, as it is also the case for fermionic loops.



$$-ig_s T_{ij}^a \gamma^\mu$$



$$-g_s f^{abc} [(p-q)^\lambda g^{\mu\nu} + (q-r)^\mu g^{\nu\lambda} + (r-p)^\nu g^{\lambda\mu}]$$



$$\begin{aligned} & -ig_s^2 [f^{eac} f^{ebd} (g^{\mu\nu} g^{\lambda\delta} - g^{\mu\delta} g^{\nu\lambda}) \\ & + f^{ead} f^{ebc} (g^{\mu\nu} g^{\lambda\delta} - g^{\mu\lambda} g^{\nu\delta}) \\ & + f^{eab} f^{ecd} (g^{\mu\lambda} g^{\nu\delta} - g^{\mu\delta} g^{\nu\lambda})] \end{aligned}$$

Figure 1.1: The physical vertices in QCD. From top to bottom: the quark-gluon-antiquark vertex, the triple gluon vertex, and the quartic gluon vertex.

### 1.3 Renormalisation

With the Feynman rules of the previous section, any QCD physical quantity can be calculated as a series expansion in positive powers of the squared coupling  $g_s^2$ . We

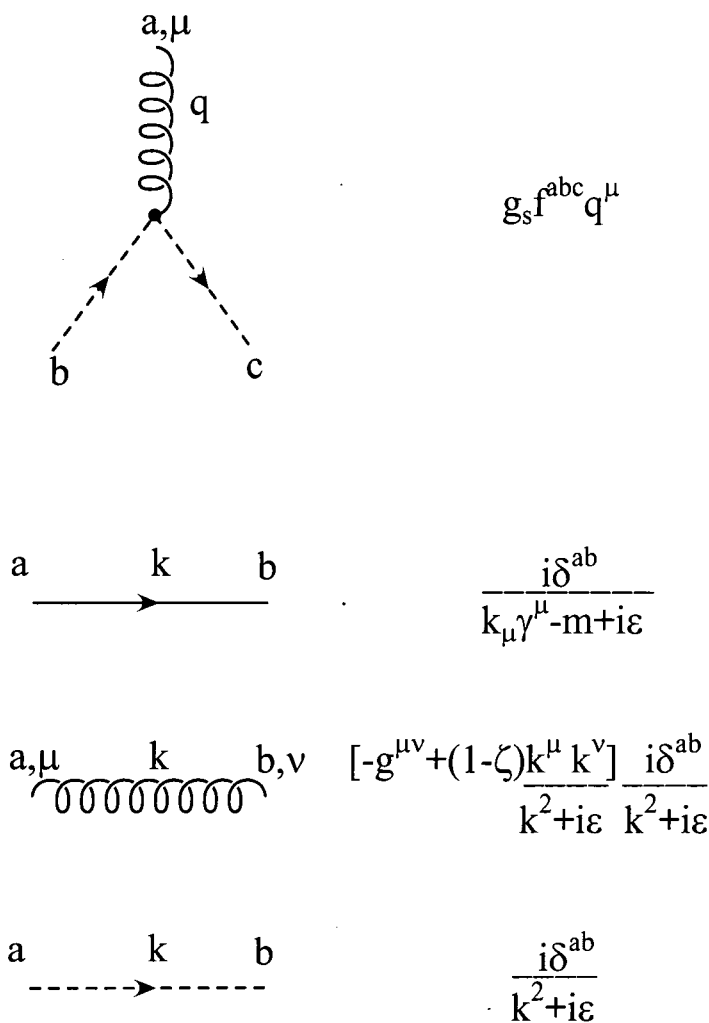


Figure 1.2: The vertex with ghosts, and (from top to bottom) the fermion, gluon and ghost propagators.

note in passing that such a series  $\left(\sum_{k=0}^{+\infty} d_k g^{k+1}, g = g_s^2\right)$  is not the most general form for the expansion of an analytical function. Also, using it relies on the assumption that the transition from  $g > 0$  to  $g = 0$ , which implies going from a world with interactions to one without, is smooth, and this is a special step. There are other subtle points to be made about the sort of power expansions that perturbative field theories generate, but we will leave that still until the next Chapter.

To calculate physical quantities in QCD perturbation theory, one starts by drawing the Feynman diagrams pertaining to a certain process, and then one calculates

each one of these with the rules provided. The tree-level or leading order diagrams (that is, all diagrams with at most two vertices) do not present special problems, and give us the first term in this series, proportional to  $g_s^2$ . However, when calculating radiative corrections to the leading order result, the relevant diagrams will have loops which include divergent internal momenta integrations. These divergences can occur either in the lower limit of integration, and they are hence known as *infrared divergences* (since they correspond physically to the low momentum region of integration), or they can occur in the higher limit of integration, and they are hence known as *ultraviolet divergences* (since this limit corresponds physically to the high momentum region of integration). There are at least three ways to deal with these divergences. One can introduce an arbitrary cut-off in the limits of integration (this has the disadvantage of spoiling Lorentz invariance); one can add in the integrand a propagator with an arbitrary, very large, mass (this is known as *Pauli-Villars regularisation* and it has the disadvantage of spoiling gauge invariance); or one can use *dimensional regularisation*, which requires assuming that the integration can be performed in an arbitrary space-time dimension  $d$ , and then analytically continued to the physical  $d=4$ . This last method will be outlined below. Its most important feature is that, by setting  $d=4-\epsilon$ , the divergences from the integrals are isolated in the form of poles in  $\frac{1}{\epsilon}$  (in the limit  $\epsilon \rightarrow 0$  one recovers the physical space-time dimensions). Once the form of the infinities is found, one can proceed to remove them by adding counterterms to the Lagrangian. The finite parts arising from the regularised integrals will dress the “bare” quantities in the original Lagrangian such as the coupling  $g_s$ , the mass of the quarks, and the gluon propagator. This last step is known as *renormalisation*. We shall now detail the calculation of the next-to-leading order (NLO) radiative corrections to the gluon propagator.

### 1.3.1 One example of regularisation at work: radiative corrections to the gluon propagator

By straightforward application of the Feynman rules, one can easily see that the radiative correction to the gluon propagator coming from the insertion of one fermionic loop ((a) in Figure 1.3) is given by

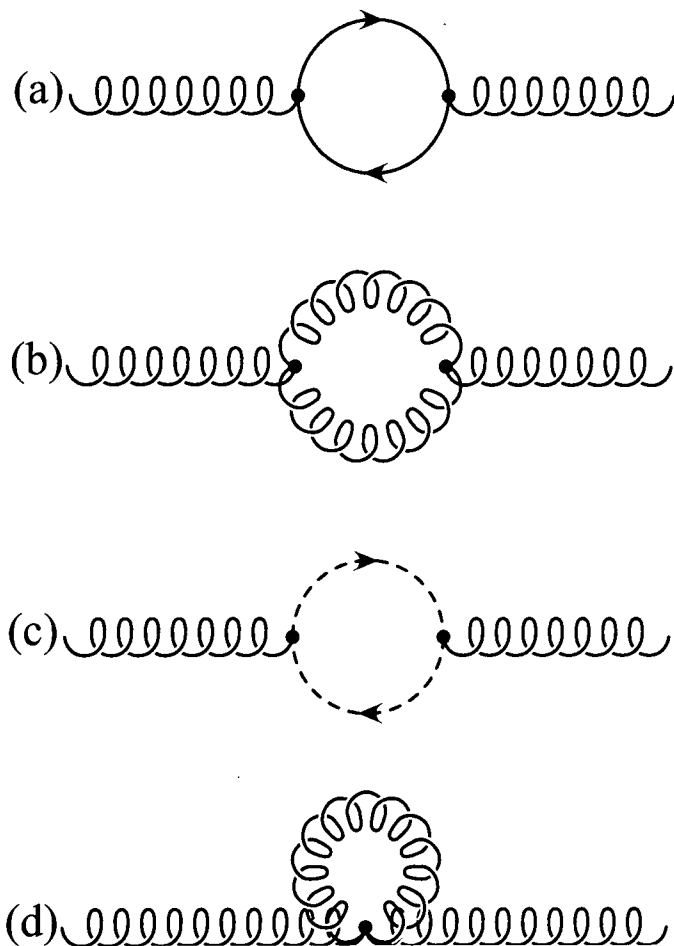


Figure 1.3: The radiative corrections to the gluon propagator at NLO: (a) fermions (quarks) loop; (b) gluons loop; (c) ghosts loop; (d) gluon loop (quartic interaction).

$$i\Pi_{\mu\nu}^{abF}(q^2) = -g_s^2 N_f \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \gamma_\mu \frac{1}{q + \not{k}} \gamma_\nu \frac{1}{\not{k}} \right] \text{Tr} [T^a T^b], \quad (1.17)$$

where a sum over the number of available quarks (hereby considered massless) was performed, and  $q$  is the external momentum. The integral to be calculated turns out to be

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k_\delta (q_\lambda + k_\lambda)}{k^2 (q+k)^2}, \quad (1.18)$$

which is logarithmically divergent. However, the integral can be performed if we allow analytical continuation in the number of space-time dimensions. So far as the integral is concerned, this amounts to changing the volume element:  $\frac{d^4 k}{(2\pi)^4} \rightarrow \mu^\epsilon \frac{d^d k}{(2\pi)^d}$ ,

where  $d=4-\epsilon$ , and the arbitrary constant  $\mu$  is introduced to keep the action dimensionless. The final result from the fermionic loop alone is<sup>2</sup>

$$\Pi_{\mu\nu}^{abF}(q^2) = -\frac{g_s^2}{8\pi^2} \frac{2}{3} N_f \delta^{ab} (g_{\mu\nu} q^2 - q_\mu q_\nu) \frac{1}{\epsilon} + \text{finite terms}. \quad (1.19)$$

The non-abelian contribution (that is, the one coming from the gluon and ghost loops<sup>3</sup>, respectively (b) and (c) in Figure 1.3) is

$$\Pi_{\mu\nu}^{abNA}(q^2) = \frac{g_s^2}{8\pi^2} \frac{5}{3} N \delta^{ab} (g_{\mu\nu} q^2 - q_\mu q_\nu) \frac{1}{\epsilon} + \text{finite terms}. \quad (1.20)$$

Thus, considering the corrections at one loop order to the gluon propagator introduces a finite correction, and an infinite, divergent (in  $\frac{1}{\epsilon}$ ) contribution of  $\mathcal{O}(g_s^2)$  to the gluon propagator. It must also be noted that, if we consider the abelian and non-abelian contributions together, we have the following total contribution at  $\mathcal{O}(g_s^2)$ :

$$\left( \Pi_{\mu\nu}^{abF} + \Pi_{\mu\nu}^{abNA} \right) (q^2) = \left( \frac{5}{3} N - \frac{2}{3} N_f \right) \frac{g_s^2}{8\pi^2} \delta^{ab} (g_{\mu\nu} q^2 - q_\mu q_\nu) \frac{1}{\epsilon} + \text{finite terms}. \quad (1.21)$$

### 1.3.2 The renormalisation group

If QCD theory were defined in terms of bare fields alone, it would, by consequence of the last subsection, be ill-defined, since it would have a divergent gluon propagator beyond leading order. In order to cancel this divergence, one will have to add a counterterm to the bare Lagrangian  $\mathcal{L}_B$  of (1.16). This can be done redefining  $G_{a,\mu} \longrightarrow \sqrt{Z_3} G_{a,\mu}^B$ , where  $Z_3 = 1 - \frac{g_s^{B^2}}{8\pi^2} (5 - \frac{2}{3} N_f) \frac{1}{\epsilon}$  to one loop order as we saw (we discard finite contributions here). As we considered before the next-to-leading order corrections to the gluon propagator, we could equally well have considered the renormalisation of the quark propagator, which amounts to renormalising the quark wave-function:  $\Psi \longrightarrow \sqrt{Z_2} \Psi^B$ , where  $Z_2 = 1 + \frac{g_s^{B^2}}{8\pi^2} \frac{4}{3} \frac{1}{\epsilon}$ . The renormalisation of the quark-gluon-antiquark vertex ( $qg\bar{q}$ ) is similarly dependent on a factor  $Z_1 =$

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<sup>2</sup>The factor in  $\frac{1}{\epsilon}$  arises from expanding the  $\Gamma$  function (see Appendix A for a definition of the  $\Gamma$  function and for a review of its properties), specifically from the expansion (A.4), which is what always appears in practice.

<sup>3</sup>The diagram (d) in Figure 1.3 is identically zero.

$1 + \frac{g_s^{B^2}}{8\pi^2} \frac{13}{3} \frac{1}{\epsilon}$ . Thus, it follows that  $g_s \rightarrow g_s^B Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} \mu^{-\frac{\epsilon}{2}}$ . Replacing the values of the renormalisation factors mentioned before, one arrives at the important result

$$g_s = g_s^B \left( 1 - \frac{g_s^{B^2}}{8\pi^2} \left( \frac{11}{2} - \frac{1}{3} N_f \right) \frac{1}{\epsilon} + \mathcal{O}(g_s^{B^4}) \right) \mu^{-\frac{\epsilon}{2}}. \quad (1.22)$$

Clearly, a two-loop calculation of the renormalisation of the coupling would bring an exact contribution at  $\mathcal{O}(g_s^{B^4})$ , a three loops calculation would bring an exact contribution at  $\mathcal{O}(g_s^{B^6})$ , and so on for higher orders in perturbation theory. These effects would be negligible in QED, where the coupling variation is very small at accessible energies ( $\alpha_{QED} \simeq \frac{1}{137}$ ), but they are not in QCD where the size of the coupling at working energies varies in the range  $0.1 - 0.4$ . Therefore, the QCD coupling is said to be a *running coupling*. To study how it runs with energy we shall now turn.

Once we are removing infinities from the Lagrangian, it is a matter of convention how much finiteness one removes as well. One possible choice is to use the *momentum subtraction* (MOM) scheme, in which a vertex (rather than a self-energy) is renormalised close to some momentum of interest ( $\mu \simeq Q$ ). This method has the disadvantage of being dependent on the vertex ( $qg\bar{q}$ ,  $ggg$ , or ghost vertices) which is chosen. Using dimensional regularisation as we did above, the renormalisation arbitrariness at one loop order resides not only in the size of the finite factor subtracted when the  $\frac{1}{\epsilon}$  pole is removed (which can be specified by a  $\tilde{\Lambda}$  parameter which will be defined below), but also in the *renormalisation scale*  $\mu$ . A full specification of the two is called choosing the *renormalisation scheme* at one loop order. When the finite factor subtracted is zero, one has the *minimal subtraction* (MS) scheme. When a factor of  $(\ln 4\pi - \gamma_E)$  is also subtracted with the  $\frac{1}{\epsilon}$  pole, one has the widely used *modified MS* ( $\overline{MS}$ ) scheme. One can relate the MOM scheme to the  $\overline{MS}$  scheme with the renormalisation scale  $\mu = e^{u+v/b} Q$ . For instance, MOM based on the  $ggg$  at  $\mu = Q$  corresponds to  $\overline{MS}$  with  $u = 2.56$  and  $v = Nf(\zeta)$ , where  $f(\zeta)$  is a third order polynomial on the gauge parameter  $\zeta$ . So, one disadvantage of the MOM scheme is its intrinsic gauge-dependence, whereas in the  $\overline{MS}$  the gauge dependence cancels order by order. The so-called *V-scheme* corresponds to  $\overline{MS}$  with  $\mu = \exp(-5/6)Q$  [7]. Nevertheless, it must be emphasised that the final, physical result should not

depend on the renormalisation scale  $\mu$ , which was introduced in purely dimensional grounds. This dependence is merely an artefact of the theoretical calculation, and would not be present in a full, nonperturbative calculation. (We shall see in section 1.4 several ways of dealing with this dependence.) In the other regularisation methods, this arbitrariness would express itself in the dependence on the cut-off, or on the arbitrary mass, which would be the arbitrary parameters introduced.

It is in the study of the dependence of physical quantities on the arbitrary scale  $\mu$  that one will find the key to understand the running of the coupling. Obviously, the physical coupling which can actually be measured will be independent of the renormalisation scale  $\mu$ . The dependence on the renormalisation scheme is just a feature of our theory. So, let us consider a generic bare Green function, to be renormalised with a  $Z$  which accounts for all the types of external fields:  $\Gamma^B(q; g_s^B, \mu_0) = Z^{-1} \Gamma(q; g_s, \mu)$ . This Green function is by definition independent of  $\mu$ ,  $\frac{\partial}{\partial \mu} \Gamma^B(q; g_s^B, \mu_0) = 0$ . Thus it follows that

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g_s) \frac{\partial}{\partial g_s} - \gamma(g_s) \right) \Gamma(q; g_s, \mu) = 0, \quad (1.23)$$

where

$$\beta(g_s) = \mu \frac{\partial}{\partial \mu} g_s \Big|_{g_s^B, \mu_0}, \quad (1.24)$$

and

$$\gamma(g_s) = \mu \frac{\partial}{\partial \mu} \ln Z \Big|_{g_s^B, \mu_0}. \quad (1.25)$$

The equation (1.23) is known as *Callan-Symanzik equation*. It describes how a change in the arbitrary renormalisation scale  $\mu$  leads to changes in the couplings and fields. Through the scale  $\mu$  ( $= \mu(Q)$ ), the Callan-Symanzik equation allows us to study the momentum dependence of the result of a QCD-theoretical calculation.

The derivation of equation (1.23) was generic. For the more specific case of QCD, one has two  $\gamma$  functions, one each for fermions and gluons:  $\gamma_F$  and  $\gamma_A$ . One also needs to account for the gauge dependence. Thus, one has

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g_s) \frac{\partial}{\partial g_s} - n_A \gamma_A(g_s) - n_F \gamma_F(g_s) + \delta(g_s) \frac{\partial}{\partial \zeta} \right) \Gamma(q; g_s, \zeta, \mu) = 0. \quad (1.26)$$



where  $n_A$  and  $n_F$  stand for the number of gluonic and fermionic fields, respectively, and

$$\delta(g_s) = \mu \frac{\partial \zeta}{\partial \mu} \Big|_{g_s^B, \mu_0}, \quad (1.27)$$

where we have chosen once again to work on a generic covariant gauge.

### 1.3.3 The $\beta$ -function equation and the one loop running coupling

From the results discussed in sub-section (1.3.1), it seems natural to expect the renormalisation group (RG)  $\beta$ -function (1.24) to have an expansion in powers of the coupling  $g_s$  of the form

$$-\beta_0 \frac{g_s^3}{16\pi^2} - \beta_1 \frac{g_s^5}{(16\pi^2)^2} - \beta_2 \frac{g_s^7}{(16\pi^2)^3} + \dots \quad (1.28)$$

Clearly,  $\beta_0 = 11 - \frac{2}{3}N_f$  (see (1.22)). Since, in physical problems, it is always the square of the  $qg\bar{q}$  vertex coupling which results from the calculations, one uses  $\alpha_s = \frac{g_s^2}{4\pi}$  more often than  $g_s$ . In this thesis, we shall use  $a = \frac{g_s^2}{4\pi^2}$ . With this convention, the RG  $\beta$ -equation (1.24) can be re-written as

$$\frac{da}{d\tau} = -a^2(1 + ca + c_2a^2 + \dots) \quad (\equiv -\beta(a)), \quad (1.29)$$

where  $\tau \equiv b \ln \frac{\mu}{\tilde{\Lambda}}$ ,  $\tilde{\Lambda}$ <sup>4</sup> has energy dimensions and it is related to the arbitrary integration constant,  $\mu$  is the renormalisation scale, and  $b = \frac{\beta_0}{2}$ ,  $c = \frac{\beta_1}{4\beta_0}$ ,  $\dots$ ,  $c_k = \frac{\beta_k}{4^k\beta_0}$ ,  $\dots$  are the RG  $\beta$ -function coefficients at one loop order, two-loop order, and so on. One is now in a position to obtain the momentum-dependence of the coupling by integration of the above equation. It turns out that, to  $n+1$ th loops order, one has (it will be detailed how this equation comes about in section 4.1)

$$\tau = \frac{1}{a} + c \ln \left| \frac{ca}{1+ca} \right| + \int_0^a \left[ \frac{1}{x^2(1+cx)} - \frac{1}{x^2B(x)} \right] dx, \quad (1.30)$$

where  $B(x) \equiv 1 + cx + c_2x^2 + \dots + c_nx^n$  is trivially related to the  $(n+1)$  loops  $\beta$ -function. Lacking, at the present state of knowledge, a nonperturbative definition

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<sup>4</sup>The  $\tilde{\Lambda}$  used in this thesis is distinct from the  $\Lambda$  of, *e.g.*, [8]:  $\tilde{\Lambda} = \Lambda \left( \frac{2\epsilon}{b} \right)^{-c/b}$ .

of the coupling, one can find it from (1.30) at any perturbative order, provided the  $\beta$ -function coefficients are known. However, the  $\beta$ -function coefficients have only been calculated to four loops order at the time of writing. We quote here their values, the calculation of  $b$  having been indicated already. The first two are universal [9, 10]:

$$b = \frac{11}{6}C_A - \frac{1}{3}N_f, \quad (1.31)$$

$$c = -\frac{7}{8}\frac{C_A^2}{b} - \frac{11}{8}\frac{C_A C_F}{b} + \frac{5}{4}C_A + \frac{3}{4}C_F, \quad (1.32)$$

but the remaining coefficients are renormalisation scheme dependent, so we quote the first two of them in the  $\overline{MS}$  scheme on which they have been calculated [11, 12]:

$$c_2^{\overline{MS}} = \frac{2857 - \frac{5033}{9}N_f + \frac{325}{27}N_f^2}{64b}, \quad (1.33)$$

$$c_3^{\overline{MS}} = \frac{3564\zeta_3 + \frac{149753}{6} - (\frac{6508}{27}\zeta_3 + \frac{1078361}{162})N_f + (\frac{6472}{81}\zeta_3 + \frac{50065}{162})N_f^2 + \frac{1093}{729}N_f^3}{256b} \quad (1.34)$$

(see (A.8) for a definition of the  $\zeta_\ell$  symbols).

The solution of the (trivial) one loop ( $c = 0, c_2 = 0, \dots, c_n = 0, \dots$ ) version of equation (1.29) is

$$a^{1-loop}(\mu) = \frac{a(\tilde{\Lambda})}{1 + a(\tilde{\Lambda})b \ln \frac{\mu}{\tilde{\Lambda}}}. \quad (1.35)$$

The constant  $\tilde{\Lambda}$  reminds us of the absence of a boundary condition for the differential equation (1.29), and thus signals the point where one loop QCD breaks down ( $\tau=0$ ) -hence known as the *Landau pole*. The constant  $\Lambda_{QCD} = \exp(-1/ba(\tilde{\Lambda}))\tilde{\Lambda}$  is more convenient. Historically, the resulting version of the one loop coupling at the physical scale  $\mu=Q$ :

$$a^{1-loop}(Q) = \frac{1}{b \ln \frac{Q}{\Lambda_{QCD}}}, \quad (1.36)$$

has been so widely used as the running coupling that it became virtually identified with it. The  $\Lambda_{QCD}$  in (1.36) is known as the *dimensional transmutation* parameter. As mentioned before, it defines the subtraction scheme, which can be the same at all orders. For instance, it can be seen [13] that  $\tilde{\Lambda}_{\overline{MS}} = \sqrt{4\pi} \exp(-\frac{\gamma_E}{2})\tilde{\Lambda}_{MS} = 2.66\tilde{\Lambda}_{MS}$ .  $\Lambda_{QCD}$  can be looked at as the QCD fundamental parameter rather than the coupling

[14, 15]. That the coupling blows up at  $Q \approx \Lambda_{QCD}$  reminds us that QCD perturbation theory becomes insufficient at low energies, a point at which some other physical theory (nonperturbative in nature) is needed to describe the *confinement* of quarks inside hadrons. One usually assumes  $\Lambda_{QCD} \approx 200$  MeV, and this energy value corresponds to 1 fm, a typical hadronic size. Opposite to what happens in QED (where the scale  $\Lambda_{QED} \approx 10^{280}$  MeV is enormous), the variation in the QCD coupling is significant at accessible energies. As mentioned before, the coupling  $a^{1-loop}(Q)$  is thus a function of the energy in practical problems. On the limit  $Q \rightarrow +\infty$ , it vanishes, a feature known as *asymptotic freedom*, and phenomenologically well understood. It tells us that QCD works better in the high-energy limit where quarks are asymptotically free.

The expansion of the coupling in equation (1.35) at the physical scale  $\mu=Q$

$$a^{1-loop}(Q) = a(\tilde{\Lambda}) \left( 1 - a(\tilde{\Lambda})b \ln \frac{Q}{\tilde{\Lambda}} + a^2(\tilde{\Lambda})b^2 \ln^2 \frac{Q}{\tilde{\Lambda}} + \dots \right), \quad (1.37)$$

shows us that the renormalised coupling can be looked at as being the effective resummation to all orders of the logarithms which arise in the renormalisation of the  $qg\bar{q}$  vertex. It thus became customary to refer to  $a^{1-loop}(Q)$  as a *renormalisation group improved coupling*.

### 1.3.4 Exact solutions of the renormalisation group $\beta$ -function equation at two-loop order

Solving equation (1.29) when truncated to two-loop order ( $c_2=0, c_3=0, \dots, c_n=0, \dots$ ) amounts to finding a physically meaningful inverse of

$$F(x) \equiv \frac{1}{x} + c \ln \left| \frac{cx}{1+cx} \right|, \quad (1.38)$$

which we define as  $G(z)$  ( $G(F(x))=x$ ). The two-loop coupling will be as universal as  $a^{1-loop}(Q)$ , because it will depend only on the scale  $\tilde{\Lambda}$ , and on the RG  $\beta$ -function coefficients  $b$  and  $c$ , which are both renormalisation scheme independent. However, a coupling  $a^{n-loops}(Q)$  with  $n \geq 3$  will be both scale and scheme dependent, as indeed will be seen in the next sub-section.

It was recently recognised [16] that the solution for the inverse of (1.38) can be explicitly given in terms of the Lambert W function (see Appendix B for the definition and properties of the Lambert W function). But, since the Lambert W function has a numerable infinity of branches, it is not *a priori* clear how to choose the branch(es) with physical meaning. To clarify the situation, we show the real-valued branches of the inverses of  $F(x)$  in Figure 1.4. The notation is as follows:

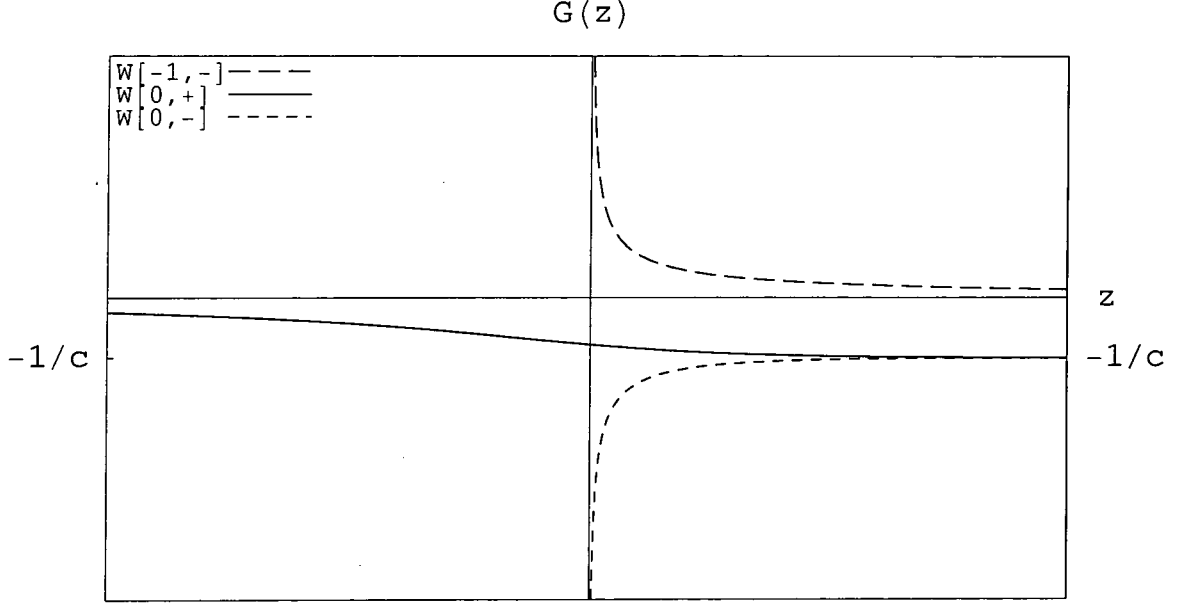


Figure 1.4: All the real-valued solutions of the two-loop  $\beta$ -function equation.

$$W[-1, -](z) = -\frac{1/c}{1 + W_{-1}(-e^{-\frac{z}{c}-1})}, \quad z \in ]0, +\infty[, \quad (1.39)$$

$$W[0, +](z) = -\frac{1/c}{1 + W_0(e^{-\frac{z}{c}-1})}, \quad z \in ]-\infty, +\infty[, \quad (1.40)$$

$$W[0, -](z) = -\frac{1/c}{1 + W_0(-e^{-\frac{z}{c}-1})}, \quad z \in ]0, +\infty[. \quad (1.41)$$

It must be noted that these three branches exhaust all possibilities. The real domains of both  $W_0(z)$  and  $W_{-1}(z)$  (the only two real branches of  $W(z)$ ) are entirely mapped into  $F(x)$ . To obtain  $G(z)$ , we note that the choice of  $G^-(z)$  is unique, but that we are faced with two branches and one semi-branch to choose  $G^+(z)$  from<sup>5</sup>. The

<sup>5</sup>We use the definitions  $G^+(z) \equiv G(z)$ ,  $z > 0$ , and  $G^-(z) \equiv G(z)$ ,  $z < 0$ .

only branch consistent with the phenomenological properties of the QCD coupling (asymptotic freedom) is actually  $W[-1, -](z)$ . So, since we shall need the full  $G(z)$  for the purposes of this thesis,  $G(z)$  is hereby defined as a function with two branches:

$$G(z) = \begin{cases} -\frac{1/c}{1 + W_{-1}(-e^{-\frac{z}{c}-1})}, & z > 0, \\ -\frac{1/c}{1 + W_0(e^{-\frac{z}{c}-1})}, & z \leq 0. \end{cases} \quad (1.42)$$

For the coupling, one only needs to consider positive values of  $z$ , and therefore the two loops coupling at the physical scale  $\mu = Q$  is

$$a^{2\text{-loops}}(Q) = -\frac{1}{c} \frac{1}{1 + W_{-1}\left(-\frac{1}{e}\left(\frac{Q}{\Lambda_{QCD}}\right)^{-\frac{b}{c}}\right)}. \quad (1.43)$$

The two-loop coupling, besides obeying asymptotic freedom as chosen above, also blows up at  $Q = \Lambda_{QCD}$ , as it happened at the one loop level. Figure 1.5 illustrates how the one loop and two-loop couplings compare. This explicit solution for the two-loop coupling has been used in the context of analytic perturbation theory [16], and also in a proposal of a new perturbative expansion using renormalisation scheme invariants [17]. In this thesis, the two-loop coupling (1.43) shall be preferred over the one loop coupling, because since the one loop coupling effectively sets  $c = 0$ , it does not include as much RG information as the two-loop coupling.

### 1.3.5 The renormalisation group $\beta$ -function equation at three loops order

The integration of the truncation to three loops order of equation (1.29) ( $c_3 = 0, c_4 = 0, \dots, c_n = 0, \dots$ ), leaves us with the following function to invert:

$$H(x) \equiv \frac{1}{x} + c \ln \left| \frac{cx}{\sqrt{1+cx+c_2^{RS}x^2}} \right| + \frac{2c_2^{RS}-c^2}{\sqrt{4c_2^{RS}-c^2}} \arctan \left( \frac{c+2c_2^{RS}x}{\sqrt{4c_2^{RS}-c^2}} \right). \quad (1.44)$$

It is not clear at the time of writing how expression (1.44) could be explicitly inverted, even only for  $x > 0$ , and therefore how a coupling exact to three loops could be obtained. We note however that the procedure followed in [16] (replacement of

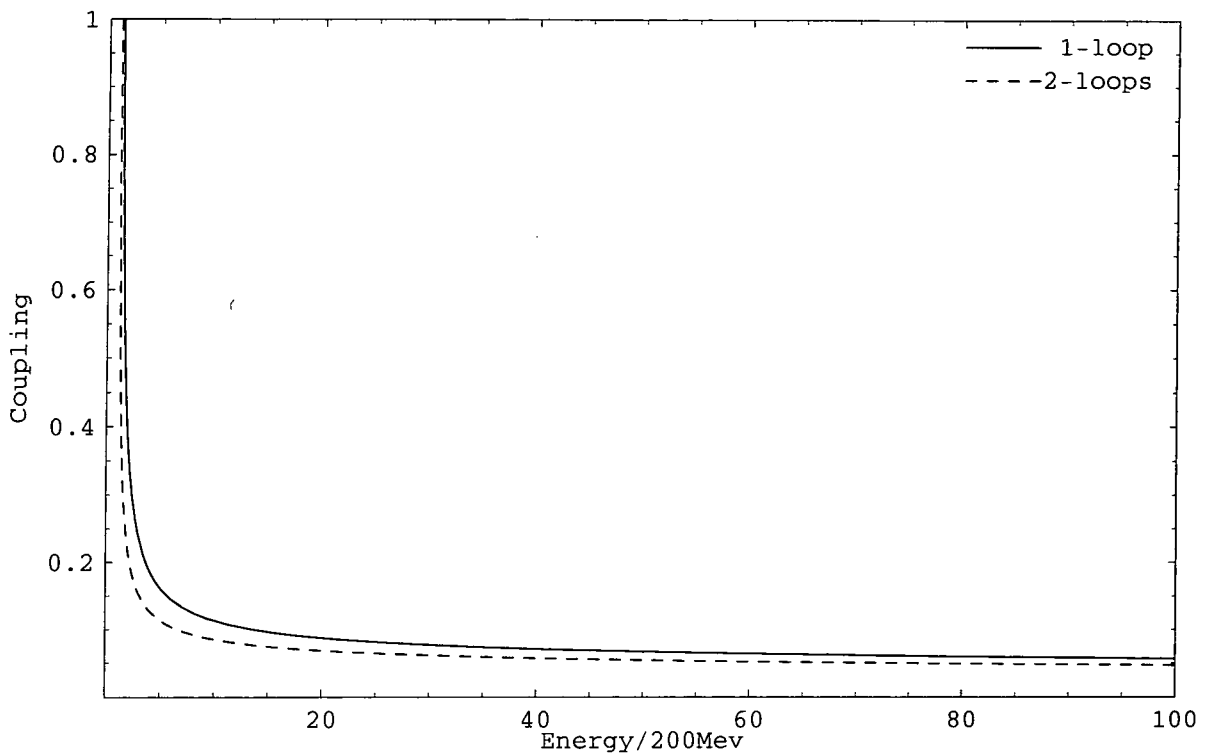


Figure 1.5: The one loop and two-loop QCD couplings with  $\Lambda_{QCD} = 200$  MeV.

the RG  $\beta$ -function equation by its  $P_1^1$  Padé approximant, leading to a solution which can be inverted with the Lambert W function) provides an invertible function which is qualitatively different (crucially, in its singularity structure) from the three loops exact  $H(x)$  given above.

We also note that the three loops coupling would be hopelessly renormalisation scheme dependent (through  $c_2^{RS}$ ), as indeed would be any of the higher orders renormalised couplings.

## 1.4 Renormalisation-scheme dependence

A generic QCD observable can always, if necessary, be divided by a constant and raised to a suitable power to be written as

$$D(a) = a + d_1 a^2 + d_2 a^3 + \dots d_k a^{k+1} + \dots, \quad (1.45)$$

without any loss of generality. However, only the LO (tree-level, leading order) perturbative coefficient is renormalisation scheme invariant. We have seen before how one loop calculations results are dependent both on a subtraction procedure and on an arbitrary constant. Thus, already at NLO one has a renormalisation scheme dependence problem. In higher orders, the subtraction procedure can presumably be chosen to be the same, but an entirely new arbitrary constant will be introduced at each order. The dependence on these arbitrary constants is obviously unphysical, and, since the all-orders  $D(a)$  should not depend on any arbitrary parameters, it is presumed that the renormalisation scheme dependence would cancel in an all-orders calculation.

This section will discuss the general problem of renormalisation scheme (RS) dependence. How to parameterize the RS dependence in a consistent and practical way, and how to deal with the unphysical dependence on the RS parameters themselves shall be its goal.

### 1.4.1 Labelling the renormalisation scheme

As we saw, both the coupling and the perturbation series coefficients beyond leading order are dependent on the renormalisation scheme used. In fact, the differential equation for the coupling lacks a boundary condition whose absence introduces the parameter  $\tilde{\Lambda}$ , and  $d_1 = d_1(\tilde{\Lambda}, \mu)$ <sup>6</sup>. This dependence can be jointly parameterized as  $\tau = b \ln \frac{\mu}{\tilde{\Lambda}}$ . Still concerning the coupling, we saw at length how its RS dependence depends on the order at which we truncate the RG  $\beta$ -function (1.29). The two-loop coupling (1.43), as well as the one loop coupling (1.36), only depend on the parameter  $\tau$  (thanks to the universality of the first two  $\beta$ -function parameters), but the three loops coupling would be dependent on the non-universal three loops coefficient of the  $\beta$ -function,  $c_2^{RS}$ . A generic  $k+1$  loops coupling would depend on  $k$  parameters:  $a^{k+1-loops}(\tau, c_2, \dots, c_k)$  ( $k > 2$ ). For consistency, the coefficients of order higher than one in QCD perturbation theory also depend on the same RS parameters [18]. Therefore, one has generally  $d_1(\tau), d_2(\tau, c_2), d_3(\tau, c_2, c_3), \dots, d_k(\tau, c_2, c_3, \dots, c_k), \dots$ . In

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<sup>6</sup>These problems do not arise in QED because there is a “natural” renormalisation point, thanks to the fact that the QED coupling has meaning in the low-energy, macroscopical world, where it can be measured as the fine-structure constant.

general, when truncating (1.45) up to and including  $d_k a^{k+1}$ , one has to truncate (1.29) up to and including  $c_k a^{k+2}$ . Thus, the  $k+1$ th order truncated approximant,

$$D^{(k+1)} = a^{(k+1)} + d_1(a^{(k+1)})^2 + \dots + d_k(a^{(k+1)})^{k+1}, \quad (1.46)$$

will have its RS-dependence labelled by the first  $k$  RS parameters,  $D^{(k+1)} = D^{(k+1)}(\tau, c_2, c_3, \dots, c_k)$ , and one then expects the differences between two schemes to be, formally, effects one order higher in perturbation theory [18], that is

$$D'^{(k+1)}(\tau', c'_2, \dots, c'_k) - D^{(k+1)}(\tau, c_2, \dots, c_k) = K(\tau', c'_2, \dots, c'_k; \tau, c_2, \dots, c_k) a^{k+3}. \quad (1.47)$$

Therefore, since the renormalisation scheme is completely characterised by a choice of the infinite sequence of parameters  $\tau, c_2, c_3, \dots, c_k, \dots$ , these parameters are said to label the RS.

### 1.4.2 The NLO renormalisation-scheme invariant

In many approaches to large orders QCD, it is of importance to find RS-invariant quantities at each order in perturbation theory. RS invariants can be built order by order from the general properties of the renormalisation group. We shall deduce here the NLO RS-invariant while leaving the general problem of higher orders to a later sub-section.

Let us compare two different RS's by defining the relation between the couplings  $a$  and  $a'$  in two schemes to be

$$a' = a + \nu_1 a^2 + \nu_2 a^3 + \dots \quad (1.48)$$

If one replaces (1.48) in a series expansion  $D(a') = a' + d'_1(\tau')a'^2 + \dots$ , we can then equate powers of  $a$  with (1.45), enabling us to conclude that

$$d_1(\tau) - d'_1(\tau') = \nu_1. \quad (1.49)$$

On the other hand, since  $\tau = \frac{1}{a}$  at NLO (1.30), one also deduces that

$$\tau - \tau' = \frac{1}{a} - \frac{1}{a'}, \quad (1.50)$$



and, equating powers of  $a$ , one has to first order from the last equation

$$\tau - \tau' = \nu_1 + \mathcal{O}(a), \quad (1.51)$$

and thus

$$d_1(\tau) - d'_1(\tau') = \tau - \tau', \quad (1.52)$$

at NLO. This implies that we have found a NLO RS-invariant quantity,

$$\tau - d_1(\tau) = \tau' - d'_1(\tau') \equiv \rho_0 \quad \left( \equiv b \ln \frac{Q}{\Lambda_R} \right). \quad (1.53)$$

It must be noted that  $\rho_0$  (or, equivalently,  $\Lambda_R$ ) is completely independent of the renormalisation procedure, although it is an observable-dependent quantity. The previous equation can also be stated at  $\mu = Q$  as

$$\Lambda_R = \tilde{\Lambda}_{RS} e^{d_1^{RS}/b}, \quad (1.54)$$

a form in which it is known as *Celmaster-Gonsalves relation* [13].

A number of procedures have been suggested to deal with the renormalisation scheme dependence of truncated calculations in QCD perturbation theory. We shall now briefly review some of these methods.

### 1.4.3 The physical scale

The approach which is probably more often taken in NLO calculations is to dispose of the renormalisation scale dependence by choosing it to be close to the physical scale of the problem,  $\mu \simeq Q$ . The motivation for this viewpoint is the fact that at each order in perturbation theory potentially large terms of the form  $a(\mu)b \ln \frac{\mu}{Q}$  arise from the loop integrations, so

$$d_k \propto \sum_{l=0}^k C_{kl} \left( b \ln \frac{\mu}{Q} \right)^l. \quad (1.55)$$

Thus, by setting  $\mu = Q$  one avoids these large logarithms. This overlooks the fact that the renormalisation scheme dependence, even at NLO, is not totally given by the renormalisation scale. A general  $d_k$  is dependent on  $k$  RS parameters,  $d_k =$

$d_k(\tau, c_2, \dots, c_k)$ , where  $\tau = b \ln \frac{\mu}{\tilde{\Lambda}}$ . Thus, the coefficients  $C_{kl}$  above will also depend on  $\tilde{\Lambda}$ . So, if we write instead

$$d_k \propto \sum_{l=0}^k \tilde{C}_{kl} (\tau - \rho_0)^l, \quad (1.56)$$

(where  $\rho_0 = \tau - d_1(\tau)$  is the NLO RS-invariant) one can use

$$\tau - \rho_0 = b \ln \left( \frac{\mu}{Q} e^{d_1^{RS}/b} \right) \quad (1.57)$$

to write (1.56) as

$$d_k \propto \sum_{l=0}^k \tilde{C}_{kl} \left( b \ln \left( \frac{\mu}{Q} e^{d_1^{RS}/b} \right) \right)^l. \quad (1.58)$$

The coefficients  $\tilde{C}_{kl}$  depend on the perturbation theory coefficients  $d_2, d_3, \dots, d_k$ , and on the RS parameters  $c_2, c_3, \dots, c_k$ , but crucially do not depend on the NLO RS choice. Choosing the physical scale can now be recognised as leading effectively to an expansion of each  $d_k$  as a polynomial in  $d_1$  ( $d_k \propto \sum_{l=0}^k \tilde{C}_{kl} d_1^l$ ).

Choosing  $\mu$  as to avoid large logarithms in (1.58) corresponds to  $\mu = e^{-d_1^{RS}/b} Q$ . This, as we shall see, is equivalent, at NLO, to the effective charge (EC) scheme where we can set  $d_1^{EC} = 0$  because  $d_1$  is absorbed into the coupling [19].

#### 1.4.4 The Principle of Minimal Sensitivity

It was suggested by P.M. Stevenson in [18] that, since the final result of a physical calculation should be independent of arbitrary, unphysical parameters, one should choose such parameters by minimising the dependence on them of the truncated approximant to the physical quantity being calculated. This was termed Principle of Minimal Sensitivity (PMS). It is most straightforwardly implemented by calculating the stationary point of a truncated perturbation series with respect to the arbitrary parameters. For instance, at a given order  $k$  in QCD perturbation theory, one has a truncated perturbation series  $D^{(k+1)} = D^{(k+1)}(\tau, c_2, c_3, \dots, c_k)$ , and the PMS point is specified by solving the set of equations

$$\left. \frac{\partial D^{(k+1)}}{\partial \tau} \right|_{\tau=\tau^{PMS}} = \left. \frac{\partial D^{(k+1)}}{\partial c_2} \right|_{c_2=c_2^{PMS}} = \left. \frac{\partial D^{(k+1)}}{\partial c_3} \right|_{c_3=c_3^{PMS}} = \dots = \left. \frac{\partial D^{(k+1)}}{\partial c_k} \right|_{c_k=c_k^{PMS}} = 0, \quad (1.59)$$

for the parameters  $\tau^{PMS}, c_2^{PMS}, c_3^{PMS}, \dots, c_k^{PMS}$ , whose values will characterise the PMS renormalisation scheme. The PMS bases itself on the philosophy that the coupling and the  $\beta$ -function are mere artifacts, and that consequently the quantity to which the PMS must be applied is the physical observable. Some alternative formulations of the PMS are at variance with this doctrine, namely the “conformal mapping” approach (see, *e.g.*, [20] and references therein), where the coupling, rather than the physical observable, is the quantity being fitted. The PMS has been used outside the field of renormalisation scheme issues, for instance in lattice QCD and in several perturbation theory problems in quantum mechanics models and other theories, such as  $\phi^4$  field theory (see, *e.g.*, [21] and references therein).

It can happen that there are several stationary points. When this occurs, either one chooses the broadest stationary point (this choice is known as “strong PMS”) or, if there is information available about the final result, one uses it to choose the physically sensible stationary point. It has been found that the opposite problem also occurs: there might be no stationary point at all at a given order, or for a sub-sequence of orders (odd or even orders, for instance). How to resolve these shortcomings of the PMS has ultimately to be dealt with case by case.

The problem of finding the PMS point at NLO in QCD perturbation theory amounts to find the stationary point of

$$D^{(2)}(a) = a(\tau) + d_1(\tau)a^2(\tau), \quad (1.60)$$

with respect to the parameter  $\tau$  alone. The solution can be found solving numerically the transcendental equation

$$\tau = \rho_0 - \frac{c}{2} \left( 1 + \frac{1}{W_{-1}(-e^{-\frac{\tau}{c}-1})} \right). \quad (1.61)$$

Defining now  $\bar{a} \equiv a(\tau^{PMS})$ , one has for the approximant

$$D_{PMS}^{(2)}(\bar{a}) = \bar{a} \frac{1 + \frac{1}{2}c\bar{a}}{1 + c\bar{a}}. \quad (1.62)$$

How to find the PMS point at a given order  $k$  in QCD perturbation theory is a problem that, despite having been addressed in the original proposal of the PMS procedure [18], has not been thoroughly investigated ever since. An attempt at using the PMS at orders  $k=1$  and  $k=2$  for the  $R$  ratio of the  $\tau$  decay showed that the PMS was not very useful in this particular case [23]. To address the problem at a generic order  $k$ , one defines, analogously to the  $\beta$ -function (1.29),

$$\beta_i(a) \equiv \frac{\partial a}{\partial c_i} = a^2(1 + ca + c_2 a^2 + \dots) \int_0^a \frac{x^{i+2}}{[x^2(1 + cx + c_2 x^2 + \dots)]^2} dx \quad (i \geq 2), \quad (1.63)$$

as the dependence of  $a$  on the RS parameters (excluding  $\tau$ , obviously). By expanding the right-hand side of the above equation, one has

$$\frac{\partial a}{\partial c_i} = \frac{1}{i-1} a^{i+1} (1 + W_1^i a + W_2^i a^2 + \dots) \quad (i \geq 2), \quad (1.64)$$

where

$$W_j^i = -\frac{1}{i+j-1} \sum_{r=1}^j (i+j-1-2r) c_r W_{j-r}^i \quad (j \geq 1) \quad (1.65)$$

( $c_1 \equiv c$ ,  $W_0^i \equiv 1$ ), and then one obtains [18]

$$\frac{\partial a}{\partial c_i} = \frac{a^{i+1}}{i-1} \left[ 1 - \frac{i-2}{i} ca + \left( \frac{(i-1)(i-2)}{i(i+1)} c^2 - \frac{i-3}{i+1} c_2 \right) a^2 + \dots \right] \quad (i \geq 2). \quad (1.66)$$

Imposing the PMS condition

$$\left. \frac{\partial D^{(k+1)}}{\partial(\tau, c_2, \dots, c_k)} \right|_{PMS} = 0, \quad (1.67)$$

one can then derive the equations [18]

$$0 = \sum_{i=0}^k a^i \sum_{m=i}^k (m+1) d_m c_{k+i-m} \quad (1.68)$$

$$\int_0^a \frac{x^{j+2}}{[\beta_{k+1}(a)]^2} dx = \frac{a^{j-1} \sum_{l=0}^{k-j} a^l \sum_{m=0}^l (m+1) d_m W_{l-m}^j}{j-1 \frac{\sum_{l=0}^{k-1} a^l \sum_{m=0}^l (m+1) d_m c_{l-m}}{\sum_{l=0}^{k-1} a^l \sum_{m=0}^l (m+1) d_m c_{l-m}}} \quad j=2, \dots, k, \quad (1.69)$$

( $d_0 \equiv 1, c_0 \equiv 1$ ). Given numerical values for  $d_1, d_2, \dots, d_k$ , the  $k-1$  equations (1.69) and equation (1.68) can be simultaneously solved for  $a, c_2, \dots, c_k$ . The solution (or solutions), if there is at least one, is (are) the PMS point(s) for  $D^{(k+1)}(a)$ .

### 1.4.5 The BLM scale

This method was proposed by S.J. Brodsky, G.P. Lepage, and P.B. Mackenzie in [7]. It consists of choosing the renormalisation scale such that all vacuum polarization effects from fermion loops are absorbed into the running coupling. This is motivated by QED, where the coupling can be defined to include all the vacuum-polarization insertions in the photon propagator. As these are the only corrections that renormalise the (slow) running QED coupling, and they vanish in the infrared limit, QED becomes a fixed-point theory at very low energies. In QCD, this technique can be implemented at each order by absorbing all the vacuum-polarization insertions from quark loops. However, being QCD a non-Abelian theory, the BLM scale depends on the process being considered.

To be explicit, let us consider the expansion to NLO of an observable in a given scheme,

$$D(a) = a + d_1 a^2. \quad (1.70)$$

In general, a NLO calculation can always be decomposed into a  $N_f$ -dependent part and a  $N_f$ -independent one:

$$d_1 = d_1^{[1]} N_f + d_1^{[0]}. \quad (1.71)$$

The BLM procedure consists of absorbing the  $N_f$ -dependent part into the BLM scale  $\mu_{BLM}$ :

$$a(\mu) \rightarrow a(\mu^{BLM}) = \frac{a(\mu)}{1 + b a(\mu) \ln \frac{\mu^{BLM}}{\mu} + \dots}. \quad (1.72)$$

Considering the expansion in the BLM scale

$$D^{BLM}(a) = a(\mu^{BLM}) + d_1^{BLM} a^2(\mu^{BLM}), \quad (1.73)$$

replacing (1.72) in (1.73), and demanding cancelation of the  $N_f$ -dependence when comparing with (1.70), leads to

$$\mu^{BLM} = \mu \exp(3d_1^{[1]}), \quad (1.74)$$

$$d_1^{BLM} = \frac{33}{2}d_1^{[1]} + d_1^{[0]}. \quad (1.75)$$

So, provided a calculation in any given scheme, the equations above determine the BLM scale and the NLO BLM coefficient.

The BLM procedure only determines the renormalisation scale, and thus it does not provide a full renormalisation prescription, even at NLO. Furthermore, it does not uniquely select the renormalisation scale, as a  $N_f$ -independent rescaling will give identical expansions in  $a(\mu^{BLM})$ . Specifically,  $\mu \rightarrow \mu' = \mu e^C$  is equivalent with the simultaneous changes  $d_1^{[1]} \rightarrow d_1'^{[1]} = d_1^{[1]} + \frac{1}{3}C$  and  $d_1^{[0]} \rightarrow d_1'^{[0]} = d_1^{[0]} - \frac{11}{2}C$ , and the two BLM expansions are equal. Despite its shortcomings, the idea at the base of BLM has inspired some interesting developments, namely the leading- $b$  expansion in QCD (sometimes termed “naive non-abelianization”) which will be discussed in section 3.2.

### 1.4.6 The effective charge scheme

The effective charge scheme proposed by G. Grunberg in [19] amounts to choosing the RS parameters such that  $d_1 = d_2 = \dots d_k = 0$ , rendering the renormalised coupling, called *effective charge*, the actual observable. The renormalisation group EC (effective charge)  $\beta$ -function is defined as

$$\beta(a) \equiv \rho(a) = a^2(1 + ca + \rho_2 a^2 + \dots \rho_k a^k + \dots). \quad (1.76)$$

As a consequence, the scheme parameters are then  $\tau^{EC} = \rho_0, c_2^{EC} = \rho_2, \dots c_k^{EC} = \rho_k, \dots$ , each  $\rho_k$  being a RS invariant (it will be seen below how RS invariants can be calculated to any order) into which all the information from the perturbative calculations and the renormalisation parameters is absorbed. In this scheme, the observable is thus the coupling itself, renormalised to  $k+1$  loops:  $D^{(k+1)}(a) = a^{(k+1)}$ . This was argued originally as a way of restoring the dependence of the observable on the physical ratio  $\frac{Q}{\Lambda}$  rather than on the coupling, which was deemed unphysical.

The EC scheme can be implemented at NLO as follows. Since one has logarithms of the type  $b \ln \frac{Q}{\mu}$  at every order in perturbation theory, one has effectively:

$$D(a(\mu)) = a(\mu) + a^2(\mu) \left( d_1^{RS} - b \ln \frac{Q}{\mu} \right) + \mathcal{O} \left( a^3(\mu) \right). \quad (1.77)$$

Now, the EC renormalisation scale is obtained by requiring the first perturbative coefficient to vanish, and this is equivalent with choosing the scale  $\mu = \exp(-d_1^{RS}/b)Q$ <sup>7</sup>. This shows that the EC scheme is equivalent, at NLO alone, to the Fastest Apparent Convergence (FAC) method, which consists of requiring the next order in perturbation theory to vanish. It must also be noted that the EC and PMS predictions are very close at NLO ( $\tau^{EC} = \rho_0$ ,  $\tau^{PMS} \simeq \rho_0 - \frac{c}{2} + \mathcal{O}(1)$ ) and NNLO [22].

Having calculated the high-order  $\rho_k$ 's, one can fit  $D^{(k+1)}(a)$  to experiment. It is then possible to extract the physical  $\rho_0$  or  $\tilde{\Lambda}$  [14, 15]. So, this scheme has the advantage of allowing a physical determination of the parameter  $\tilde{\Lambda}$ , arguably a more fundamental one than the coupling, whereas in the physical scale approach, arbitrary bounds, for instance  $\mu = \frac{1}{2}Q$  and  $\mu = 2Q$ , are postulated, providing arbitrary error bars for the theoretical calculations.

### 1.4.7 Renormalisation-scheme invariants

To find RS invariants in orders higher than NLO we choose to work in the EC scheme where  $D(a') = a'$ , and where the  $\beta$ -function is (1.76). Then, by noting that

$$\beta'(a') = \frac{da'}{da} \beta(a), \quad (1.78)$$

we can write

$$\rho(D) = \left( \frac{da(D)}{dD} \right)^{-1} \beta(a(D)), \quad (1.79)$$

where  $a(D)$  is the reversed series of the perturbation series (1.45) (see Appendix C for an explanation on how to revert power series). We can now expand both sides in (1.79) as a power series in  $D$  and equate coefficients. We obtain a sequence of RS invariants

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<sup>7</sup>Or, which is the same, replacing  $\tilde{\Lambda} \rightarrow \tilde{\Lambda} \exp(d_1^{RS}/b)$ .

$$\begin{aligned}
\rho_1 &= c, \\
\rho_2 &= c_2 + d_2 - cd_1 - d_1^2, \\
\rho_3 &= c_3 + 2d_3 - 4d_1d_2 - cd_1^2 - 2\rho_2d_1 + 2d_1^3, \\
&\vdots
\end{aligned} \tag{1.80}$$

It must be emphasised that these objects, being RS-independent, have physical significance, whereas its individual constituents, being RS-dependent:  $d_1^{RS}, d_2^{RS}, \dots$ ;  $c_2^{RS}, c_3^{RS}, \dots$ , have ultimately no physical meaning. One can rearrange these equations so that the perturbation theory expansion of  $D(a)$  is re-expressed as:

$$\begin{aligned}
d_1 &= d_1 \\
d_2(d_1, c_2) &= d_1^2 + cd_1 + \rho_2 - c_2, \\
d_3(d_1, c_2, c_3) &= d_1^3 + \frac{5}{2}cd_1^2 + (3\rho_2 - 2c_2)d_1 + \frac{1}{2}(\rho_3 - c_3), \\
&\vdots
\end{aligned} \tag{1.81}$$

This has the advantage of expressing the perturbative coefficients as functions of (the usually known)  $d_1$ , the RS-invariants (which one expects to have physical meaning), and the RG coefficients at each order (which are process independent). A variant of the above consists of recognising that because  $d_1 = \tau - \rho_0$  one can write

$$\begin{aligned}
d_1(\tau) &= \tau - \rho_0 \\
d_2(\tau, c_2) &= (\tau - \rho_0)^2 + c(\tau - \rho_0) + \rho_2 - c_2, \\
d_3(\tau, c_2, c_3) &= (\tau - \rho_0)^3 + \frac{5}{2}c(\tau - \rho_0)^2 + (3\rho_2 - 2c_2)d_1 + \frac{1}{2}(\rho_3 - c_3), \\
&\vdots
\end{aligned} \tag{1.82}$$

showing explicitly the full RS dependence. This alternative also serves to emphasise that whether one parameterizes the NLO RS-dependence by  $d_1$  or  $\tau$  is simply a matter of choice. As can be seen from (1.80), this is also true at any order  $k$  in perturbation theory: the renormalisation scheme is uniquely defined by  $d_k$  as good as  $c_k$ .



### 1.4.8 An alternative set of RS-invariants

Obviously, infinitely many other renormalisation scheme invariant combinations of perturbative coefficients can be found by adding RS-invariant pieces to each order. We shall be concerned in this sub-section by a slightly different approach from the one in the previous sub-section, but which will be of crucial importance later in this thesis.

Since the sum of a perturbation theory series expansion must be independent of the RS used, one can assume that the total derivatives with respect to the RS parameters  $\tau, c_2, c_3, \dots c_k, \dots$  vanish:

$$\frac{dD(a)}{d\tau} = \frac{dD(a)}{dc_2} = \dots \frac{dD(a)}{dc_k} = \dots = 0. \quad (1.83)$$

Using the  $\beta$ -function equation, the total derivative with respect to the first RG parameter can be written as

$$\frac{dD(a)}{d\tau} = -a^2(1+ca+c_2a^2+\dots)(1+2d_1a+3d_2a^2+\dots) + \frac{\partial d_1}{\partial \tau}a^2 + \frac{\partial d_2}{\partial \tau}a^3 + \frac{\partial d_3}{\partial \tau}a^4 + \dots \quad (1.84)$$

For the other total derivatives, we will also need to consider the derivatives with respect to the other RS parameters, and for these we use (1.66). One obtains a sequence of equations similar to (1.84) for each RS parameter, each of which must be identically zero. Since we have effectively a power series in  $a$  in each of these equations, it follows that each coefficient of a given power of  $a$  in each power series has to be zero. The first few equations that result are<sup>8</sup>:

$$\frac{\partial d_2}{\partial d_1} = 2d_1 + c, \quad \frac{\partial d_2}{\partial c_2} = -1, \quad (1.85)$$

$$\frac{\partial d_3}{\partial d_1} = 3d_1^2 + 5cd_1 + 3X_2 - 2c_2, \quad \frac{\partial d_3}{\partial c_2} = -2d_1, \quad \frac{\partial d_3}{\partial c_3} = -\frac{1}{2}, \quad (1.86)$$

$$\vdots \quad \vdots \quad \dots$$

The first column comes from the equation obtained with the derivative  $\frac{d}{d\tau}$ , the second from  $\frac{d}{dc_2}$ , ... , the first line corresponds to  $\mathcal{O}(a^3)$ , the second to  $\mathcal{O}(a^4)$ , ... . (The

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<sup>8</sup>We write here the derivatives with respect to  $\tau$  as derivatives with respect to  $d_1$  (consider 1.53).

coefficients  $d_2, d_3, \dots d_k$  are eliminated at each order  $k+1$  in favour of the constants of integration  $X_2, X_3, \dots X_k$  obtained in previous orders.) By simple integration of these equations, one finds

$$\begin{aligned} d_2(d_1, c_2) &= d_1^2 + cd_1 + X_2 - c_2 \\ d_3(d_1, c_2, c_3) &= d_1^3 + \frac{5}{2}cd_1^2 + (3X_2 - 2c_2)d_1 + X_3 - \frac{c_3}{2} \\ &\vdots \end{aligned} \quad (1.87)$$

with the general structure

$$d_k(d_1, c_2, \dots c_k) = X_k + \hat{d}_k(d_1, c_2, \dots c_k) - \frac{c_k}{k-1}. \quad (1.88)$$

The  $X_k$ 's encode the "RG-unpredictable" content at each order in perturbation theory. The remaining terms are "RG-predictable". The RS invariants  $X_k$  are distinct from the  $\rho_k$ 's discussed before. It can be easily verified that:  $X_2 = \rho_2$ ,  $X_3 = \frac{1}{2}\rho_3$ ,  $X_4 = \frac{1}{3}\rho_4 + \frac{5}{3}\rho_2^2 - \frac{1}{6}c\rho_3$ ,  $X_5 = \frac{1}{4}\rho_5 + 2\rho_2\rho_3 - \frac{1}{6}c\rho_4 - \frac{1}{12}c\rho_2^2 + \frac{1}{12}c^2\rho_3, \dots$ . To know the numerical value of a given  $X_k$  (or a  $\rho_k$ ) for a given observable, one needs a full calculation at order  $k$ , which is achieved once we know  $d_1, d_2, d_3, \dots d_k$  and  $c_2, c_3, \dots c_k$ .

### 1.4.9 Complete Renormalisation Group Improvement

All the solutions to the RS-dependence problem reviewed so far have been based on truncating the perturbative series and choosing some scale  $\mu = xQ$ . The dependence on the scale  $\mu$ , which manifests itself via logarithms  $\ln \frac{\mu}{\Lambda_{RS}}$  which are present in all the perturbative orders, is thus replaced by a dependence on the energy  $Q$ . This dependence is not the physical one, which is expected to be build by ultraviolet logarithms of the form  $\ln \frac{Q}{\Lambda_R}$ , containing the energy  $Q$  and the RS-independent parameter  $\Lambda_R$ . Furthermore, besides both having energy dimensions, there is no reason why  $\mu$  should be directly related to  $Q$ . An all-orders procedure based on the assumption that *all* the RG-predictable terms  $d_1(\mu), c_2, c_3, \dots$  must be resummed has been proposed recently by C.J. Maxwell in [2]. It will be briefly reviewed in this sub-section.

Firstly, we separate the RS-dependent and RS-independent parts of the perturbation theory series with coefficients given by (1.87) in the following way

$$D(a) = a + d_1 a^2 + (d_1^2 + c d_1 - c_2) a^3 + (d_1^3 + \frac{5}{2} c d_1^2 + (3X_2 - 2c_2) d_1 - \frac{1}{2} c_3) a^4 + \dots \\ X_2 a^3 + X_3 a^4 + \dots \quad (1.89)$$

The subset of terms in the first line which are known at NLO or only depend directly on the RS parameters is

$$a + d_1 a^2 + (d_1^2 + c d_1 - c_2) a^3 + (d_1^3 + \frac{5}{2} c d_1^2 - 2c_2 d_1 - \frac{1}{2} c_3) a^4 + \dots (\equiv a_0). \quad (1.90)$$

To know the sum of this set of terms,  $a_0$ , one notes first that, since terms dependent on the RS invariants  $X_2, X_3, \dots$  cannot cancel the RS-dependence of these terms, the total sum  $a_0$  must therefore be RS-independent in itself. So we can choose a separate RS for  $a_0$ , namely by imposing  $d_1 = 0$ , and  $c_k = 0$  for all  $k \geq 2$ . The later choice is known as the *'t Hooft scheme*, and the former implies (from (1.53))  $\tau = \rho_0$ . The use of a *'t Hooft scheme* renders the  $\beta$ -function rather simple ( $\beta(a) = a^2(1 + ca)$ ), and thus enables us to identify  $a_0$  with the two-loops coupling at the EC scale  $\mu = \exp(-d_1^{RS}/b)Q$ :

$$a_0(Q) = - \frac{1/c}{1 + W_{-1}\left(-\frac{1}{e} \left(\frac{Q e^{-d_1^{RS}/b}}{\Lambda_{RS}}\right)^{-b/c}\right)}. \quad (1.91)$$

Given a full NNLO calculation ( $d_1^{RS}, d_2^{RS}, c_2^{RS}$ ),  $X_2$  can be determined, and we have another known subset of terms which can be resummed

$$X_2 a^3 + 3X_2 d_1 a^4 + \dots = X_2 a_0^3. \quad (1.92)$$

As it can be easily understood, the RS-independence of this subset of terms guarantees once again that their sum will be proportional to  $a_0$ . By applying this reasoning at each order, one arrives at the result

$$D(a_0) = a_0 + X_2 a_0^3 + X_3 a_0^4 + \dots X_k a_0^{k+1} + \dots \quad (1.93)$$

This perturbation theory series is just the same as in (1.87) with  $d_1 = c_2 = c_3 = \dots c_k = \dots = 0$ . As it was noted already in the original work [2], because the  $X_k$ 's are defined as constants of integration, their choice is not unique.

Obviously, the procedure described above is not a complete solution of the RS dependence problem. It does however serve to eliminate any dependence on the renormalisation scale  $\mu$ , since if all  $UV$  logarithms are resummed together with logarithms involving  $\mu$ , the dependence on it cancels between the one-loop coupling (1.35) and the logarithms of  $\mu$  from the perturbative coefficients. In standard RG-improvement only a subset of the  $UV$  logarithms are resummed. This will become clear when the application of CORGI (Complete Renormalisation Group Improvement) in the context of the leading- $b$  approximation will be considered in detail in section 4.2. In this light, the CORGI approach provides a physical motivation for the choice of the EC scale.

The CORGI method was used in the study of moments of leptonproduction structure functions [24]. A similar, albeit different perturbation theory construction using the two-loop coupling and another set of RS invariants was recently proposed in [17].

# Chapter 2

## Large-Orders in Perturbation Theory

In Field Theory, it is very often assumed that physical observables are analytic functions and, also, that these analytic functions can be reconstructed by means of a perturbation series in positive powers of the coupling:

$$R(g) = \sum_{k=0}^{+\infty} r_k g^k. \quad (2.1)$$

Furthermore, it is in general promptly assumed that the series on the right-hand side converges to a unique, unambiguous result. The aim of this Chapter is to question these assumptions.

### 2.1 Why perturbation theory will diverge

#### 2.1.1 Dyson's argument for the divergence of QED perturbation theory

The widespread belief that QED perturbation series would always converge was first challenged by F.J. Dyson in [25]. He argued that there is a singularity at the origin of the coupling constant complex plane, and that, as a consequence, power

expansions in the coupling are not analytical<sup>1</sup> in the complex plane, and are expected to diverge.

Let us assume that (2.1) represents a QED perturbation series (so  $g = \frac{e^2}{4\pi}$ , “ $e$ ” being the electron charge) for a generic physical observable. After mass and charge renormalisation, the coefficients  $r_k$  in such a series are finite. If one assumes that (2.1) converges for some  $e^2 > 0$ , then  $R(e^2)$  is an analytic function of  $e^2$  as  $e^2 \rightarrow 0^+$ . For correspondingly small values of  $-e^2$ ,  $R(-e^2)$  will also be expected to be an analytic function with a convergent power series. Now, it is well-known that, in the physical world (where  $e^2 > 0$ ), the interaction between two like charges (at a distance  $r$  from each other) is repulsive in nature, say

$$e^2 \delta_+(r^2) \tag{2.2}$$

(where  $\delta_+$  is a function taking positive values). In a fictitious world with quantities given by  $R(-e^2)$ , like charges would attract and, by analogy, the interaction between them would be

$$-e^2 \delta_+(r^2). \tag{2.3}$$

In other words, the sign of the Coulomb-type potential would be reversed in this mirror-universe. So, being the energy of a system consisting of a large number of  $N$  charged particles, with mean kinetic and potential energies  $T$  and  $V$ , given by

$$E \approx TN + \frac{1}{2}e^2VN^2, \tag{2.4}$$

(the difference between a system consisting of  $N$  positrons or  $N$  electrons would amount to an overall minus sign), the ground state corresponds to  $N = 0$ , and the energy increases monotonously with  $N$ . On the other hand, in the fictitious world, we would have

$$E \approx TN - \frac{1}{2}e^2VN^2, \tag{2.5}$$

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<sup>1</sup>Technically, if a given function taking real or complex values has a convergent power series expansion in the vicinity of a given point, we say such function is *analytic* at the given point. If it is analytic everywhere in a finite region of the complex plane, we say that it is *entire*. If it is analytic everywhere in such a region except for isolated poles, we say that it is *meromorphic*.

and this would lead to the unusual property that  $E$  would decrease after a certain critical value of  $N$ :

$$\frac{\partial E}{\partial N} = 0 \Rightarrow N_{crit.} = \frac{T}{Ve^2}, \quad (2.6)$$

and therefore, for  $N$  much bigger than  $N_{crit.}$ , one would have:

$$E \sim -N^2, \quad (2.7)$$

which means that the “vacuum” state of such fictitious world ( $E \approx 0$ ) would not be the state of lowest energy. This would have dramatic consequences, as the barrier at  $N_{crit.}$  could be penetrated by quantum tunnel effect and, once a large number of  $e^+/e^-$  pairs had been created, one can imagine that electrons and positrons would be brought together in different regions of space<sup>2</sup>, leading to a never-ending decrease in the energy. Thus, the “vacuum” of this fictitious world would be unstable, and it would lead to an explosive creation of pairs of particles. Dyson concluded [25] that  $R(e^2)$  could not be analytical for any  $e^2 \leq 0$ , and that, as a consequence, the QED perturbation series could not be convergent for a non-zero value of  $e^2$ . However, as pointed out by P.M. Stevenson in [26], the non-analyticity of a function does not necessarily imply non-convergence of its perturbative series expansion. It may simply signify that part of the function is not expandable in positive powers of the coupling, and that as such it is nonperturbative in character. For instance, one may consider

$$R(g) = \frac{1}{1-g} + e^{-\frac{1}{g}}, \quad (2.8)$$

where the  $e^{-\frac{1}{g}}$  term has a power expansion in negative powers of  $g$  only, and as a consequence cannot be reproduced neither by a finite nor by an infinite sum of perturbative terms. The perturbation series expansion of  $R(g)$  converges inside a circle in the complex plane which extends until the singularity located at  $|g| = 1$ , and reproduces the other term<sup>3</sup>. In fewer words, this example shows us that the

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<sup>2</sup>One assumes a sufficiently low density of charged particles, and a region of space large enough for the classic Coulomb potential to be valid.

<sup>3</sup>The perturbation series in this case is simply the geometrical series  $1 + g + g^2 + \dots + g^k + \dots = \frac{1}{1-g}$  (provided  $|g| < 1$ ).

*summability* of the perturbation series does not imply that the physical observable is fully *recoverable* from its perturbation series.

### 2.1.2 Extension of Dyson's argument to include fermions

Dyson's argument as stated in the previous subsection is only valid for charged bosons. It can be extended to fermions as outlined below [27, 28]. Denoting by  $k$  the fermion momentum, the kinetic energy of  $N$  fermions in  $d-1$  euclidean space dimensions can be written as

$$K.E. \approx \int |k| d^{d-1}k. \quad (2.9)$$

Since we expect the volume element to be

$$\int d^{d-1}k \sim N, \quad (2.10)$$

and we expect the Fermi sphere to be filled up to  $|k| \propto \rho(x)^{\frac{1}{d-1}}$  (where  $\rho(x)$  is the fermionic local density), it follows that

$$K.E. \approx N^{1+\frac{1}{d-1}}. \quad (2.11)$$

We note in passing that for  $d \rightarrow \infty$  we would recover the bosonic result. The total energy for  $N$  fermions is then

$$E \approx T N^{1+\frac{1}{d-1}} + gVN^2. \quad (2.12)$$

We can calculate  $N_{crit.}$  in the same way as before:

$$\frac{\partial E}{\partial N} = 0 \Rightarrow N_{crit.} = \left( \frac{T}{2V} \frac{d}{d-1} \frac{1}{g} \right)^{\frac{d-1}{d-2}}. \quad (2.13)$$

Whereas we had  $E_{crit.} \sim g^{-1}$  for charged bosons, we now have  $E_{crit.} \sim g^{-\frac{d}{d-2}}$ . So, we conclude that, if consideration of the Pauli principle seems to render the vacuum decay more difficult, nonetheless it does not undo the fact that we have an essential singularity at the origin.



### 2.1.3 Counting Feynman diagrams: $\phi^4$ field theory as an example

The last two subsections suggest that the radius of convergence for any perturbation theory expansion will be zero, but the arguments presented are of a semiclassical nature. Nevertheless, the existence of an essential singularity can be proved, from field theory principles, in the special case of QCD [29]. The existence of this singularity suggests that, in field theory, the growth of the large-order terms may be very strong, perhaps even factorial. However, the growth in the number of diagrams alone might be a sufficient cause for divergence. An attempt, using statistical physics methods, at calculating the number of Feynman diagrams at order  $k$  when  $k \rightarrow \infty$  for a generalized quantum anharmonic oscillator was carried out in [30]. An asymptotic estimate for a  $\phi^4$  theory can be obtained in the following way.

Consider the partition function for a one-dimensional  $\phi^4$  field theory:

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\phi^2} e^{-g\phi^4} d\phi. \quad (2.14)$$

The integrand has an essential singularity at  $g = 0$ . This reflects the fact that replacing  $g \rightarrow -g$  implies going from a situation where the “potential” has one single, absolute minimum, to one where the “potential” is not bounded from below. If we expand the second exponential and carelessly exchange the order of integration and summation, we end up with an expression that can be easily integrated in terms of the  $\Gamma$  function (A.1):

$$Z(g) = \sum_{k=0}^{+\infty} Z_k g^k, \quad (2.15)$$

$$Z_k = \frac{(-1)^k}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\phi^{4k}}{k!} e^{-\phi^2} d\phi = (-1)^k \frac{\Gamma(2k + \frac{1}{2})}{\sqrt{\pi} k!}. \quad (2.16)$$

The asymptotic formula for the function  $\Gamma(k)$  (A.2), and the Stirling approximation for large  $k$  (A.6) help us to conclude that

$$Z_k \simeq \frac{k^{-1}}{\sqrt{\pi}} (-4)^k k!. \quad (2.17)$$

The growth of the  $Z_k$ 's is thus stronger than a factorial, and these numbers count the number of vacuum diagrams for  $\phi^4$  field theory. Trivial as this example seems,

its results are not qualitatively different from what can be obtained in more rigorous grounds. For instance, in [31], the asymptotic estimate

$$Z_k \simeq k^\gamma z_i^k k! \left[ A + \mathcal{O}\left(\frac{1}{k}\right) \right] \quad (2.18)$$

was obtained using methods of statistical physics to count the number of Feynman diagrams in the same field theory. This method leaves the values of  $\gamma$  and  $A$  uncalculated (however, even with other methods  $A$  is not calculable). The case of a Yukawa-type theory is discussed in [28]. The case of QED is briefly discussed in [5]. We will postpone until the next Chapter a thorough discussion of the (divergent) large-order behaviour of QCD. However, all estimates for both the number of diagrams and the actual values of large order coefficients reduce to the most general case (2.18) (see [32] for a recent review containing a comprehensive set of references, and [33] for a review concerning QCD). So, we will have to face the fact that the increase in the large-order perturbation theory coefficients will generally be strong enough to outgrow the effect of taking higher powers of the expansion parameter.

## 2.2 Asymptotic series

The observations made in the previous section make it clear that unless we are ready to drop the common-sensical assumption that physical observables ought to be represented by analytic functions, one has to weaken the assumption that observables ought to be represented in the whole complex domain by convergent series in positive powers of the coupling. In fact, we saw how perturbation theory alone will never give us an *unique* reconstruction of an observable. Moreover, we also saw that perturbation theory series may well be divergent. In the following, one will assume that, in general, (2.1) may diverge for all  $g \neq 0$ . However, a divergent perturbation series may still be assumed to be asymptotically convergent.

**Definition 1 : Asymptotic series** *A power series is said to be asymptotic to a function  $R(g)$  analytic on the set  $\mathcal{A}$  (symbolically  $R(g) \sim \sum_{k=0}^{+\infty} r_k g^k$ ) if, for all natural  $N$  there is a  $g \in \mathcal{A}$  such that:*

$$\left| R(g) - \sum_{k=0}^N r_k g^k \right| \leq |r_{N+1} g^{N+1}|; \quad (2.19)$$

$\mathcal{A}$  being a subset of the neighbourhood of the origin in the complex plane, defined such that  $|\arg(g)| \leq \frac{\pi}{2}$  and such that  $|g|$  is finite.

Thus, a series is said to be asymptotic to a function if the (absolute value of the) remainder after summing any  $N$  terms is bounded by the (absolute value of the) next non-zero term in the series. The difference with convergent series is obviously that the remainder is not required to vanish when  $N \rightarrow \infty$ . As a consequence, when one speaks of asymptotic convergence to a given function, this convergence is meant within a certain accuracy. In fact, whereas a convergent series converges to a unique function with infinite accuracy, an asymptotic series is asymptotic to a whole class of functions, *i.e.* it converges modulus a (sub-dominant) function. (It must be noted that the converse is not true: each function has a unique asymptotic series expansion.) Also, whereas the domain of convergence for a convergent series is a fixed region of space, for an asymptotic series the value of the expansion parameter is related to the optimal number of terms to sum. In fact, for a fixed  $g$ , the remainder of the series decreases until a certain  $k_{opt.}$ , and then it starts to increase. Therefore, it is of interest to know the optimal number of terms. So, let us assume that the series coefficients at large  $k$  grow like

$$r_k = A k^\gamma z_i^k k!. \quad (2.20)$$

In order to find  $k_{opt.}$ , one minimizes the remainder with respect to  $k$ , and solves the resulting equation:

$$\left. \frac{\partial}{\partial k} (r_k g^k) \right|_{k=k_{opt.}} = 0 \Rightarrow k_{opt.} \simeq -\frac{\gamma + \frac{1}{2}}{W_0[-|gz_i|(\gamma + \frac{1}{2})]}. \quad (2.21)$$

The only approximation used here was the Stirling approximation for large  $k$ , and, since  $W_0$  is the principal branch of the Lambert W function (see Appendix B), (2.21) holds provided  $|gz_i|(\gamma + \frac{1}{2}) \leq \frac{1}{e}$ . This result enables us to calculate  $k_{opt.}$  for any perturbative field theory where the general form of the coefficients can be written as in (2.20). It must be noted that the estimate of the optimum number

of terms to be summed depends mostly on the factor  $|gz_i|$ . The reason why the dependence on the  $\gamma + \frac{1}{2}$  almost factors out can be understood using the expansion of  $W_0$  around  $g=0$  (B.2) to obtain:

$$k_{opt.} \simeq \frac{1}{gz_i + (\gamma + \frac{1}{2})(gz_i)^2 + \frac{3}{2}(\gamma + \frac{1}{2})^2(gz_i)^3 + \mathcal{O}(g^4)}. \quad (2.22)$$

So we can see that, for small enough values of  $g$ ,  $k_{opt.}$  is dominated by a linear term in  $g$ .

When summing an asymptotic series, the best approximation occurs when the series is terminated at its minimal term, and a rough estimate of the error can be given by a function of  $g$  defined as an upper bound in the accuracy of the partial sum of  $k$  terms:

$$\epsilon(g) \equiv \left| \min_{\{k\}} (r_k g^k) \right| = |r_{k_{opt.}} g^{k_{opt.}}|. \quad (2.23)$$

Using the Stirling approximation and (2.21), it follows that

$$\epsilon(g) \simeq |A| \sqrt{2\pi} \left( \frac{\exp(W_0[-|gz_i|(\gamma + \frac{1}{2})]) \exp(1/W_0[-|gz_i|(\gamma + \frac{1}{2})])}{|gz_i|e} \right)^{\gamma + \frac{1}{2}}. \quad (2.24)$$

For small values of  $g$ , the expansion (B.2) is valid and we have  $\exp(W_0[-|gz_i|(\gamma + \frac{1}{2})]) \sim \mathcal{O}(1)$ , and consequently the leading behaviour is approximately given by the second exponential:

$$\epsilon(g) \simeq |A| \sqrt{2\pi} \frac{e^{-k_{opt.}}}{(g|z_i|e)^{\gamma + \frac{1}{2}}} \sim e^{-\frac{1}{g|z_i| + \dots}}. \quad (2.25)$$

The question which must now be addressed is how can we associate a function with a given asymptotic series. In some cases, this can be done using the Borel transform method.

### 2.2.1 The Borel method of resummation

**Definition 2 : Borel summation** *A given series (say, of the form (2.1)) is Borel summable if:*

(i) its Borel transform  $B[R](z)$ ,

$$B[R](z) = \sum_{k=0}^{+\infty} \frac{r_k}{k!} z^k, \quad (2.26)$$

converges inside a circle of radius  $|z| < \delta$  ( $\delta > 0$ ),

(ii)  $B[R](z)$  can be analytically continued to an infinite strip of non-vanishing width bisected by the positive real semi-axis ( $\text{Re}(z) \geq 0$ ),

(iii) and the inverse of (2.26), the integral

$$\mathbf{B}(g) \equiv \int_0^{+\infty} e^{-z} B[R](gz) dz, \quad (2.27)$$

called the Borel sum, exists for some  $g \neq 0$ .

By construction (A.7),

$$\sum_{k=0}^{+\infty} r_k g^k = \sum_{k=0}^{+\infty} \frac{r_k}{k!} \int_0^{+\infty} e^{-z} (gz)^k dz \sim \mathbf{B}(g), \quad (2.28)$$

i.e. the original series is asymptotic to the Borel sum. It must be noted that it was assumed that the order of the integration and summation operations can be changed, and this is only valid if the original series has a non-vanishing radius of convergence  $\rho$ . If the radius of convergence is zero or if we are interested in values of  $z$  outside the circle of convergence, the Borel sum must be taken as a *redefinition* of the original series.

It is of importance to have a criterion for the Borel summability of a given series. This is provided by the *Nevanlinna Theorem* [32].

**Theorem 1 : Borel summability** *If  $R(g)$  is an analytic function in a region  $\mathcal{K}(\eta)$  defined such that  $\text{Re}(\frac{1}{g}) > \frac{1}{\eta}$  ( $\eta > 0$ ),  $\text{Re}(g) \geq 0$ , and the origin is included (see Figure 2.1), and if  $R(g)$  has the asymptotic expansion*

$$R(g) \sim \sum_{k=0}^{+\infty} r_k g^k, \quad (2.29)$$

and if, furthermore, for some positive constants  $C_1$  and  $C_2$  the remainder is bounded in such a way that

$$|\epsilon(g)| < C_1 C_2^N N! |g|^N \quad (2.30)$$

uniformly for all  $g \in \mathcal{K}(\eta)$ , and all  $N$  above some  $N_0$ , then  $R(g)$  can be represented by the Borel sum for any  $g \in \mathcal{K}(\eta)$ .

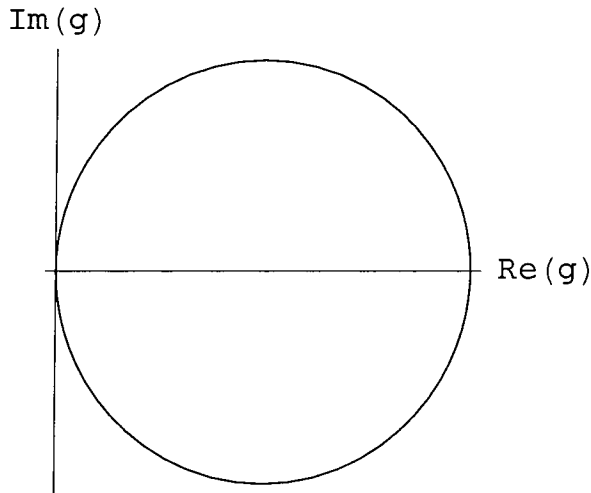


Figure 2.1: The domain  $\mathcal{K}(\eta)$  where the Nevanlinna Theorem applies.

So, we see that we can only resum non-ambiguously a divergent series at the price of requiring an extended domain of analyticity for the function that we are trying to reconstruct.

The Borel method shall be valued as useful to obtain the singularity structure of the original series. This comes about because the convergence properties of (2.26) are better than those of (2.1), as can be seen by comparing the respective radiuses of convergence:

$$\frac{1}{\rho_1} = \lim_{k \rightarrow +\infty} \sup \sqrt[k]{|r_k|},$$

$$\frac{1}{\rho_2} = \lim_{k \rightarrow +\infty} \sup \sqrt[k]{\frac{|r_k|}{k!}}.$$

Clearly, if  $\rho_1$  is non-vanishing, then  $\rho_2$  will be infinite. Thus, singularities condensed at the origin will spread out to the complex plane by taking the Borel transform.

As a first example of an asymptotic series that can be Borel resummed, one may consider an alternating-sign factorial series such as

$$R(g) = \sum_{k=0}^{+\infty} (-1)^k k! g^k, \quad (2.31)$$

which has  $\rho_1 = 0$  (*i.e.*, it diverges for every  $g \neq 0$ ). Anyhow, a meaning can be assigned to the sum of this series. Its Borel transform is simply

$$B[R](z) = \sum_{k=0}^{+\infty} (-z)^k = \frac{1}{1+z} \quad (\text{for } |z| < 1), \quad (2.32)$$

and the Borel sum is<sup>4</sup>

$$B(g) = \int_0^{+\infty} \frac{e^{-z}}{1+gz} dz. \quad (2.33)$$

Fortunately, the singularity in the integral above sits outside the range of integration, and therefore the integral exists. In general, this may not be the case. For instance, with a slightly more general behaviour for the growth of the coefficients:

$$r_k \propto z_i^{-k} k!, \quad (2.34)$$

we also have a simple Borel transform

$$B[R](z) \propto \sum_{k=0}^{+\infty} \left( \frac{z}{z_i} \right)^k = \frac{1}{1 - \frac{z}{z_i}} \quad (\text{for } |z| < |z_i|), \quad (2.35)$$

but we have instead a Borel sum which may not exist,

$$B(z) \propto \int_0^{+\infty} \frac{e^{-z}}{1 - \frac{gz}{z_i}} dz, \quad (2.36)$$

because if  $z_i > 0$  we will have a singularity at  $z = \frac{z_i}{g}$  which will prevent the convergence of the integral for any positive value of the “coupling”  $g$ . In that case, one has to define a prescription to go around the pole. This can be done by shifting the pole from the real axis:

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<sup>4</sup>This result can also be expressed as  $B(g) = -\frac{\epsilon}{g} Ei(-\frac{1}{g})$ , where  $Ei(x)$  is the exponential integral function (A.9). Alternatively, the same result can be expressed as  $B(g) = \frac{\epsilon^{1/g}}{g} E_1(\frac{1}{g})$ , where  $E_n(x)$  is the generalised exponential integral function (A.10).

$$B(g) \propto \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} \frac{e^{-z}}{1 - \frac{gz}{z_i} \pm i\epsilon} dz \pm i\pi e^{-\frac{z_i}{g}}, \quad (2.37)$$

(the choice of the sign depends on whether one integrates above or below the positive semiaxis). The limit in (2.37) exists although the integral has no meaning, and it is called the *Cauchy Principal Value*. The residue (the imaginary part) gives us a measure of the ambiguity in the Borel sum. This ambiguity is  $\propto e^{-\frac{z_i}{g}}$  and as such it is heavily suppressed for small values of the “coupling” constant  $g$ .

On the other hand, if the large-order behaviour is assumed to be

$$r_k \propto k^\gamma z_i^{-k} k! \quad (2.38)$$

(as it is usually the case in field theories of physical interest) one will have a branch point instead of a pole in the denominator of the Borel sum:

$$B(g) \propto \int_0^{+\infty} \frac{e^{-z}}{(1 - \frac{gz}{z_i})^{1+\gamma}} dz. \quad (2.39)$$

This can be seen working backwards, using the following identity (which is related to the Gauss’s hypergeometric function):

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \frac{z^{\alpha-1} e^{-z}}{(1 + gz)^\beta} dz = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} \frac{z^{\beta-1} e^{-z}}{(1 + gz)^\alpha} dz. \quad (2.40)$$

With this identity in hand one can clearly see that

$$\int_0^{+\infty} \frac{e^{-z}}{(1 - \frac{gz}{z_i})^{1+\gamma}} dz = \frac{1}{\Gamma(1+\gamma)} \int_0^{+\infty} \frac{e^{-z} z^\gamma}{1 - \frac{gz}{z_i}} dz. \quad (2.41)$$

The fraction in the integrand can now be expanded, and the series thus obtained is recognised as one which involves  $\Gamma$  functions. It is then enough to use the Stirling approximations in order to recover the original series:

$$B(g) \propto \sum_{k=0}^{+\infty} \Gamma(k + \gamma + 1) \left(\frac{g}{z_i}\right)^k \simeq \sum_{k=0}^{+\infty} k^\gamma z_i^{-k} k! g^k. \quad (2.42)$$

The ambiguity in the prescription needed to define the Borel sum of  $R(g)$  for this most general case is

$$\delta R(g) \propto \left(\frac{z_i}{g}\right)^\gamma \frac{e^{-\frac{z_i}{g}}}{\Gamma(1+\gamma)}, \quad (2.43)$$



and a simple numerical comparison proves that  $\epsilon(g) > \delta R(g) \forall g > 0$ , with  $\epsilon(g) \gg \delta R(g)$  for  $g \ll 1$ . Thus, the ambiguity arising from a non-exact Borel sum is always less than the “error” implied in truncating the same series at its minimal term.

### 2.2.2 The balance between the resummation method and the analyticity domain

In the previous subsection, a method to select uniquely a function from a general, but divergent, asymptotic series was described. However, the conditions that guarantee the convergence of the Borel sum, stated in the Nevanlinna theorem, aside from requiring a bound in the remainder, also demand a domain of analyticity (and boundedness) which might be in excess of the domain of the function being reconstructed. In fact, as QED (and arguably, also QCD [29, 34]) has a singularity at the origin, the opening angle at the origin must be zero, whereas the Nevanlinna theorem requires an opening angle at the origin of  $\pi$ . In fact, in [29] it was proved that the analyticity domain of the Green functions of QCD is a horn-shaped region with zero opening angle at the origin (see Figure 2.2). So, it would be interesting to have a generalisation of the Borel procedure with convergence assured in such a region. This can be achieved by the following *Theorem* of Moroz [35].

**Theorem 2 : Summability of the Moroz sum** *Let  $R(g)$  be meromorphic in the wedge  $\mathcal{W}$  with boundary  $|F(\frac{1}{g})| = F(\frac{1}{\rho})$  (see Figure 2.2) for  $F(g) = \sum_{k=0}^{+\infty} \frac{g^k}{\mu(k)}$ , with*

$$\mu(k) = \int_0^{+\infty} e^{-e^z} z^k dz; \quad (2.44)$$

*and let also  $R(g)$  be continuous up to the boundary. If the remainder satisfies the condition*

$$|\epsilon(g)| \leq C\mu(N)|g|^N \quad (2.45)$$

*uniformly for all  $g \in \bar{\mathcal{W}}$  and for every  $N$  ( $C$  is some positive constant), then the generalisation of the Borel transform defined by*

$$M[R](z) = \sum_{k=0}^{+\infty} \frac{r_k}{\mu(k)} z^k, \quad (2.46)$$

converges for  $|z| < 1$ , and  $R(g)$  is uniquely represented by the absolutely convergent integral

$$R(g) = \int_0^{+\infty} e^{-e^z} M[R](gz) dz, \quad (2.47)$$

for any  $g \in ]0, \rho[$ .

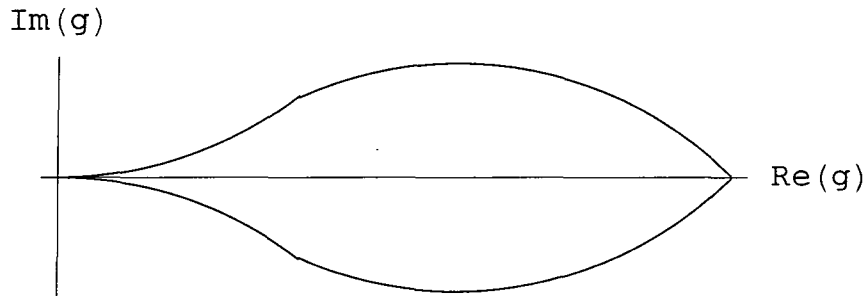


Figure 2.2: The domain  $\mathcal{W}$  where the Moroz Theorem applies. This is also the general form for the expected domain of analyticity of QCD.

Theorem 2 shows us that the Borel method has nothing special, and that many other generalisations of the Borel method could be used to resum divergent series. In the case presented here,  $\mu(k)$  has a slower growth than  $k!$  at large  $k$ <sup>5</sup>. However, one could imagine yet another generalisation of the Borel transform, for instance with  $\mu(k) = (k!)^2$ . Nevertheless, most field theories have a leading growth of the coefficients dominated by  $k!$  [32], and this is why the Borel method will be of importance in part of the following. We shall not be worried about the fact that the convergence of the Borel integral will not be assured for QCD, since the Nevanlinna theorem is just a sufficient condition of convergence, and the Borel sum may still exist outside its domain of assured convergence.

## 2.3 Padé approximants

The divergence of perturbation series mentioned in previous sections indicates the presence of singularities, or rather the inadequacy of series in positive powers

---

<sup>5</sup> $\mu(k) \simeq \sqrt{\frac{2\pi \ln k}{k}} e^{-\frac{k}{\ln k}} \left(\ln \frac{k}{\ln k}\right)^k$  [35].

of the expansion parameter to deal with such singularities. Padé approximants, being quotients of polynomials, are well-suited to model singularities, in particular poles. They also have the advantage, when compared to any of the resummation methods of the Borel type, of not requiring the knowledge of *all* terms in a given series, consequently enabling us to study its usefulness even when only the first few coefficients of a perturbation series are known. One further advantage, of great importance to QCD, is that diagonal Padé approximants are known to reduce the renormalisation scheme dependence, at least in the case of the Polarised Bjorken Sum Rule [36, 37], and become exactly renormalisation scale independent in the one-loop limit where only  $b$  is considered in the RG  $\beta$ -function [38].

**Definition 3 : Padé approximants**    *The  $P_M^N(g)$  Padé approximant to a function  $R(g)$  is the rational function*

$$P_M^N(g) \equiv \frac{\sum_{i=0}^N p_i g^i}{1 + \sum_{i=1}^M q_i g^i}, \quad (2.48)$$

where the  $N+M+1$  coefficients  $p_i$  and  $q_i$  are uniquely chosen by imposing that the Taylor expansion, truncated to  $\mathcal{O}(g^{N+M+1})$ , of the quotient above matches the first  $N+M+1$  terms of (2.1).

A general  $P_M^N(g)$  can be obtained from the perturbation series coefficients  $r_k$  by expansion of the following determinants [39]:

$$P_M^N(g) = \frac{\begin{vmatrix} r_{N-M+1} - r_{N-M+2}g & \cdots & r_N - r_{N+1}g & r_{N-M+1}g^{-M} \\ \vdots & \ddots & \vdots & \vdots \\ r_N - r_{N+1}g & \cdots & r_{N+M-1} - r_{N+M}g & r_N g^{-M} \\ -r_{N-M+1}g^{N+1} & \cdots & -r_N g^{N+1} & \sum_{k=0}^{N-M} r_k g^k \end{vmatrix}}{\begin{vmatrix} r_{N-M+1} - r_{N-M+2}g & \cdots & r_N - r_{N+1}g \\ \vdots & \ddots & \vdots \\ r_N - r_{N+1}g & \cdots & r_{N+M-1} - r_{N+M}g \end{vmatrix}}. \quad (2.49)$$

### 2.3.1 Padé approximants as continued fractions

The calculation of Padé approximants using determinants can be computationally cumbersome. In fact, all coefficients have to be recomputed when going from one order to the next. However, if we restrict ourselves to the *normal*<sup>6</sup> sequence of diagonal and off-diagonal Padé approximants  $P_0^0(g)$ ,  $P_1^0(g)$ ,  $P_1^1(g)$ ,  $P_2^1(g)$ ,  $P_2^2(g)$ ,  $\dots$ ,  $P_M^N(g)$ , a more convenient, equivalent way of dealing with Padé approximants consists of representing the Padé approximants in the form of *continued fractions*  $F_0(g)$ ,  $F_1(g)$ ,  $F_2(g)$ ,  $F_3(g)$ ,  $\dots$ ,  $F_n(g)$ :

$$F_n(g) = \frac{K_0}{1 + \frac{K_1 g}{1 + \frac{K_2 g}{\ddots \frac{K_{n-1} g}{1 + K_n g}}}}. \quad (2.50)$$

One simple algorithm to calculate the  $K_j$  coefficients for the continued fractions can be constructed [40] by observing that at every step  $j$  in a continued fraction one has

$$\frac{A_k^j}{B_k^j} = \frac{\frac{A_0^j}{B_0^j}}{1 + g \frac{A_k^{j+1}}{B_k^{j+1}}}, \quad (2.51)$$

and as a consequence we can write

$$A_k^{j+1} = \frac{B_{k+1}^j}{B_0^j} - \frac{A_{k+1}^j}{A_0^j}, \quad (2.52)$$

$$B_k^{j+1} = \frac{A_k^j}{A_0^j}. \quad (2.53)$$

Feeding the equations above with the initial conditions  $A_k^0 = r_k$ ,  $B_0^0 = 1$ ,  $B_k^0 = 0$ , one can identify the continued fraction coefficients simply as

$$K_j = A_0^j. \quad (2.54)$$

---

<sup>6</sup>We shall assume that every Padé in the sequence exists and that no two Padé approximants are identically equal.

If we restrict ourselves to the even truncations of (2.1), the *associated* continued fractions form (or Jacobi form) is

$$F_{2m}(g) = \frac{K_0}{1 + K_1 g - \frac{K_1 K_2 g^2}{1 + (K_2 + K_3)g - \frac{K_3 K_4 g^2}{1 + (K_4 + K_5)g - \dots - \frac{K_{2m-2} K_{2m-1} g^2}{1 + (K_{2m-1} + K_{2m})g}}}}, \quad (2.55)$$

which generates the *diagonal* sequence of Padé approximants  $P_0^0(g)$ ,  $P_1^1(g)$ ,  $P_2^2(g)$ ,  $P_3^3(g)$ ,  $\dots$ ,  $P_N^N(g)$  only.

### 2.3.2 Singularities and the convergence of Padé approximants

One question that very naturally springs to mind is which conditions are required for Padé approximants to converge. Of course, this question is independent from the convergence properties of the original power series, since practice shows that there are functions which have divergent Taylor series expansions but one, or more than one, convergent sequences of Padé approximants. Also, in many cases, the Padé approximants radius of convergence is bigger than the one of the power series. Thus, it happens for many series, either convergent or asymptotically convergent, that a better approximation may be achieved with only a few Padé approximants than by considering a great number of power series terms. One such case is the function  $\frac{1}{1+g}$ . It can be reproduced by an *infinite* sum of terms from its power series expansion, but its  $P_1^1(g)$  Padé approximant already converges exactly. This example is surely trivial, but it serves to emphasize that any function with poles or other singularities should be more easily approximated by use of Padé approximants than by use of its power series expansion. It is therefore to be expected that any meromorphic function  $R(g)$  will have a convergent sub-sequence of diagonal Padé approximants to its power series, in the domain obtained by removing the interiors of

small circles with centers at the poles<sup>7</sup>. (In practice, one must look at the extraneous zeros in the denominators of the Padé approximants. If they move away from a given region when  $N \rightarrow +\infty$ , then convergence is expected within that region.)

Of course, one may ask what happens if there are *essential*<sup>8</sup> singularities. As an example, one may consider again the function  $\exp\left(-\frac{1}{1+g}\right)$ , which has an essential singularity at  $g = -1$  that cannot be reproduced with infinite accuracy by any finite combination of terms of the form  $\frac{1}{(1+g)^n}$ . However, a neighborhood can be defined outside which the essential singularity can be modeled, to a given accuracy, by the high-order poles of the Padé approximants. In such cases, it is found that the poles of successive Padé approximants cluster about the essential singularity.

One general result concerning convergence in the continued fraction representation is the following Theorem [41].

**Theorem 3 : Convergence of Continued fractions** *Given a sequence of (real) continued fraction coefficients  $K_j$ , if the respective continued fractions sequence converges for any nonzero  $g$ , then either*

$$\sum_{j=1}^{+\infty} \left| \frac{K_2 K_4 \dots K_{2j}}{K_3 K_5 \dots K_{2j+1}} \right| \quad \text{or} \quad \sum_{j=1}^{+\infty} \left| \frac{K_3 K_5 \dots K_{2j-1}}{K_4 K_6 \dots K_{2j}} \right| \quad (2.56)$$

*must diverge.*

We note in passing that this Theorem, besides being only a necessary condition of convergence, tells us nothing about the limit function of a convergent sequence of continued fractions.

### 2.3.3 Some examples of Padé approximants at work

The content of the previous sub-sections will be clearer if we look at some examples. For instance, let us consider again series (2.31), which diverges for all non-zero  $g$ 's. The continued fraction coefficients have in this case a very simple form:  $K_0 = 1$ , and  $K_{2j-1} = K_{2j} = j$  for  $j \geq 1$ . As a consequence, Theorem 3 tells us that the successive continued fractions may converge. In fact, we shall see in the next section that,

<sup>7</sup>See Conjectures 1 and 2 of Chapter 13 in [39].

<sup>8</sup>For an essential singularity, we have neither  $\lim_{g \rightarrow c} |g - c|^\beta |R(g)| = \infty$  nor  $\lim_{g \rightarrow c} |g - c|^\beta |R(g)| = 0$ , for any positive integer  $\beta$ .



$P_M^N(g)$	Poles	Zeros
$P_2^1$	-2.618 -0.382	-0.5
$P_2^2$	-0.724 -0.276	-1.707 -0.293
$P_3^2$	-4.791 -0.5 -0.209	-0.789 -0.211
$P_3^3$	-1 -0.371 -0.159	-2.405 -0.436 -0.159
$P_4^3$	-7.3 -0.616 -0.287 -0.129	-1.069 -0.303 -0.129
$P_4^4$	-1.313 -0.444 -0.218 -0.106	-3.1 -0.573 -0.22 -0.106
$P_5^4$	-10.056 -0.746 -0.35 -0.174 -0.091	-1.345 -0.389 -0.174 -0.091
$P_5^5$	-1.648 -0.512 -0.27 -0.141 -0.079	-3.794 -0.708 -0.278 -0.141 -0.079

$P_M^N(g)$	Poles	Zeros
$P_2^1$	-1.618 0.618	0.5
$P_2^2$	-3.303 0.303	0.293 1.707
$P_3^2$	-1.93 0.213 1.217	0.211 0.789
$P_3^3$	-2.636 0.159 0.477	0.159 0.436 2.405
$P_4^3$	-2.093 0.129 0.311 1.987	0.129 0.303 1.069
$P_4^4$	-2.454 0.106 0.221 0.664	0.106 0.22 0.573 3.1
$P_5^4$	-2.182 0.091 0.175 0.41 2.923	0.091 0.174 0.389 1.345
$P_5^5$	-2.38 0.079 0.141 0.282 0.865	0.079 0.141 0.278 0.708 3.794

Table 2.1: Locations of the poles and zeros of the Padé approximants to the alternating-sign factorial series (left), and locations of the poles and zeros of the Padé approximants to the fixed-sign factorial series (right).



around the pole of the Borel integral is arbitrary<sup>9</sup>.

$P_M^N(g)$	$UV_1+IR_1+UV_2+IR_2$	
	Poles	Zeros
$P_2^1$	-1.037	
	0.269	0.267
$P_2^2$	-0.8	-1.671
	1.669	1.056
$P_3^2$	-1.227	
	-0.473	-0.491
	0.65	0.608
$P_3^3$	-1.676	
	-0.567	-0.628
	0.533	0.515
		5.997
$P_4^3$	-1.26	
	-0.497	-0.522
	0.144	0.144
	0.686	0.638
$P_4^4$	-0.899	-2.181
	-0.368	-0.371
	0.33	0.329
	2.044	1.24
$P_5^4$	-1.383	
	-0.57	-0.622
	-0.219	-0.219
	0.26	0.26
	0.819	0.741
$P_5^5$	-2.035	
	-0.657	-0.789
	-0.257	-0.257
	0.236	0.236
	0.652	0.619
		5.994

$P_M^N(g)$	$UV_1+UV_2+IR_2$	
	Poles	Zeros
$P_2^1$	-1.183	
	-0.317	-0.333
$P_2^2$	-2.045	
	-0.419	-0.486
		5.486
$P_3^2$	-1.235	
	-0.352	-0.377
	0.177	0.177
$P_3^3$	-0.655	-1.302
	-0.215	-0.216
	1.711	1.248
$P_4^3$	-1.461	
	-0.442	-0.508
	-0.153	-0.153
	0.458	0.454
$P_4^4$	-0.998	-4.658
	-0.367	-0.39
	-0.135	-0.135
	0.642	0.621
$P_5^4$	-1.521	
	-0.463	-0.541
	-0.172	-0.173
	-0.078	-0.078
	0.489	0.484
$P_5^5$	-3.106	
	-0.544	-0.731
	-0.209	-0.209
	-0.097	-0.097
	0.41	0.408
		5.477

Table 2.2: Locations of the poles and zeros of the Padé approximants to the two series corresponding to  $UV_1+IR_1+UV_2+IR_2$  (left), and  $UV_1+UV_2+IR_2$  (right).

The concrete problem, as can be seen by looking at Table 2.1, is that the poles

<sup>9</sup>However, it has been noted ([40], problem 8.59) that the Padé approximants can be made to converge to the Cauchy principal value of the Borel sum if the real part of their values with  $g \rightarrow g+i\epsilon$  is taken at each step (and  $\epsilon \rightarrow 0$ ), provided that the weight in the integral over the positive real axis is chosen positive.

cluster near to the origin on the positive side of the real line, mimicking the essential singularity there. It is expected that they will become dense as  $N \rightarrow \infty$ , making delicate any attempt to evaluate the Padé approximants at small values of  $g$ . To make matters worse, the zeros of the numerators also cluster near the origin, seeming to cancel (or nearly-cancel) the zeros of the denominators. Therefore, the Padé approximants can be very unstable when  $g$  takes values close to a pole, or a zero, or, even worse, both. This feature was already noted when Padé approximants were first studied as means of estimating higher-order terms from lower-order ones in the context of field theories [36]. One way to deal with this problem then suggested was to take first a Borel transform of the series in question, and then calculate Padé approximants to the Borel transform. It is found [36, 43] that the poles of these Padé approximants cluster around the singularities of the Borel transform, reproducing poles exactly when these exist in a finite number.

The question that now naturally arises is what happens when we consider combinations of the  $k!$  and  $(-1)^k k!$  behaviours. Do we have poles on both real semi-axes? If so, do they become dense? For instance, let us consider the following combination:

$$r_k = \left( \frac{8}{9} \left( \frac{1}{2} \right)^k + \frac{4}{9} \left( -\frac{1}{2} \right)^k - \frac{5}{18} \left( \frac{1}{4} \right)^k - \frac{1}{18} \left( -\frac{1}{4} \right)^k \right) k!. \quad (2.58)$$

The continued fraction coefficients for this case (that we will designate by  $UV_1 + IR_1 + UV_2 + IR_2$  for reasons that will be clear in the next Chapter), can be found on Table 2.3. There is a recognisable pattern, with alternate pairs of positive and negative  $K_j$ 's, thus showing the presence of fixed-sign factorial and alternating-sign factorial growths. Accordingly, one expects the poles (and zeros) to be on both sides of the real line. This is indeed what happens, as can be seen on the left-hand side of Table 2.2.

Now, let us consider a combination with a predominant Stieltjes component, say

$$r_k = \left( (-1)^k + \left( -\frac{1}{2} \right)^k + \left( \frac{1}{2} \right)^k \right) \frac{k!}{3}. \quad (2.59)$$

(We will call this combination  $UV_1 + UV_2 + IR_2$ .) The continued fraction coefficients are positive more often than they are negative, and as a consequence there will be more poles and zeros for  $g < 0$  than for  $g > 0$ . We therefore expect the Padé approximants to be more well-behaved than in the previous case. Nevertheless, the

presence of poles on the positive side of the real axis cannot be avoided once we have a fixed-sign factorial in the general, large-order  $r_k$ , irrespective of its relative “weight”. As a matter of fact, if the leading factorials in  $r_k$  are an alternating-sign factorial  $(-1)^k k!$  and a sub-leading fixed-sign factorial, say  $(1/m)^k k!$  ( $m > 1$ ), the continued fraction coefficients  $K_j$  will have, in general, a sign pattern with  $2m$  positive signs followed by two negative signs, repeated in periods of  $2(m+1)$ . This behaviour can be disturbed by extremely unequal overall factors, *i.e.*  $r_0^{UV} \gg r_0^{IR}$  or  $r_0^{UV} \ll r_0^{IR}$ . Specifically, this may delay the onset of the asymptotical periodic behaviour. To illustrate this point, we consider the case  $r_k = \left((-1)^k + 0.01\right) k! / 1.01$  in the last column of Table 2.3. The first seventeen continued fraction coefficients for all the sequences discussed in this sub-section are shown in the aforementioned Table.

## 2.4 Stieltjes series and Stieltjes functions

The results of the last section suggest that some asymptotic series may have a convergent sequence of Padé approximants, while others probably may not. It is not clear if the limit chosen by a sequence of Padé approximants will be the same as the Borel sum, and neither is it clear if there are cases in which this can be guaranteed. The concept of Stieltjes series will help clarify this situation.

**Definition 4 : Stieltjes series** *We have a Stieltjes series when there is a real, non-negative function  $\rho(z)$  ( $z \in [0, +\infty[$ ), so rapidly decreasing when  $z \rightarrow +\infty$  that all its moments*

$$r_k = (-1)^k \int_0^{+\infty} z^k \rho(z) dz \quad (k = 0, 1, 2, \dots), \quad (2.60)$$

*are finite, and in this case we call the series*

$$\sum_{k=0}^{+\infty} r_k g^k, \quad (2.61)$$

*a Stieltjes series.*

$K_j$ 's , $r_k$ 's	$(-1)^k k!$	$k!$	$UV_1 + IR_1$ $+UV_2 + IR_2$	$UV_1 +$ $UV_2 + IR_2$	$\left((-1)^k + \frac{0.01}{1.01}\right) k! / 1.01$
$K_0$	1	1	1	1	1
$K_1$	1	-1	-0.167	0.333	0.98
$K_2$	1	-1	-3.58	2.67	1.06
$K_3$	2	-2	3.4	-1.13	1.73
$K_4$	2	-2	0.741	-4.88	2.89
$K_5$	3	-3	-0.908	7.59	0.45
$K_6$	3	-3	-6.06	1.69	43.2
$K_7$	4	-4	5.88	2.31	-41.5
$K_8$	4	-4	1.69	9.82	-0.97
$K_9$	5	-5	-1.86	-4.58	3.73
$K_{10}$	5	-5	-8.28	-7.91	14.8
$K_{11}$	6	-6	8.1	14.3	-12.6
$K_{12}$	6	-6	2.75	3.67	-7.35
$K_{13}$	7	-7	-2.92	3.98	9.65
$K_{14}$	7	-7	-10.3	17.8	11.7
$K_{15}$	8	-8	10.2	-8.93	-9.21
$K_{16}$	8	-8	3.89	-10.5	-17.7

Table 2.3: Continued fraction coefficients for all the sequences discussed in subsection 2.3.3.

Every Stieltjes series is asymptotic to a *Stieltjes function*:

$$\sum_{k=0}^{+\infty} r_k g^k \sim R(g) = \int_0^{+\infty} \frac{\rho(z)}{1+gz} dz, \quad (2.62)$$

in the conditions required in Definition 1, and this can be seen by expanding the integral on the right-hand side. An example of a Stieltjes series and the respective Stieltjes function was already given in equations (2.31) and (2.33). The  $\rho(z)$  was, obviously,  $e^{-z}$ . We note in passing that, in that simple case, the integral representation of the Stieltjes function coincided with the Borel sum.

The reason to study Stieltjes series is that many useful results have been found for this type of series. For instance, it has been proved that a function has all the coefficients  $K_j$  of its continued fractions expansion non-negative if, and only if, the function being expanded is a Stieltjes function [40]. It is also a Theorem (Theorem 5.2.1 of [41]) that for a Stieltjes series all the singularities of the normal sequence Padé approximants are simple poles on the negative real line. Therefore, the convergence of the continued fractions sequence is assured for positive  $g$ . To know if the limit chosen by the continued fractions is unique, one *sufficient* condition is given by the *Carleman condition* (a proof is given in [39]): if

$$\sum_{k=1}^{+\infty} r_k^{-\frac{1}{2k}} \rightarrow \infty, \quad (2.63)$$

then  $\rho(z)$  is determined uniquely. One can easily see that for  $r_k = k^\gamma z_i^{-k} k! \simeq k^{\gamma+\frac{1}{2}} (\frac{k}{e z_i})^k$ , one has  $r_k^{-\frac{1}{2k}} \approx k^{-\frac{1}{2}} k^{-\frac{\gamma+1/2}{2k}}$ . Since  $k^{-\frac{1}{k}} \rightarrow \mathcal{O}(1)$ , the Carleman condition is satisfied and the limit of a very generic Stieltjes series is thus unique<sup>10</sup>. This leads us to Theorem 5.5.1 of [41].

**Theorem 4 : Convergence of Padé approximants to Stieltjes series** *Let (2.1) be a Stieltjes series satisfying Carleman's condition. Then the normal sequence of Padé approximants (that is, the one generated by the truncated continued fractions) converges to the Stieltjes function  $R(g)$  in a region of the complex plane with a finite radius and excluding the negative real semi-axis and the origin.*

<sup>10</sup>It is possible to prove this result for a much stronger growth of the coefficients. Indeed, if one chooses  $r_k = k^\gamma z_i^{-k} (k!)^\alpha$ , one easily arrives at the conclusion that the Carleman condition is satisfied as long as  $\alpha \leq 2$ , allowing a growth as strong as  $(k!)^2$ .

It only remains to be proved that such limit will be the Borel sum. So, let us consider again the most general case for  $r_k$ , (2.38), whose Borel sum is (2.39). As was easily seen by use of (2.40), it is possible to transform this Borel sum into an hypergeometric function:

$$\frac{1}{\Gamma(1+\gamma)} \int_0^{+\infty} \frac{e^{-z} z^\gamma}{1 - \frac{gz}{z_i}} dz = {}_2F_0(1, 1+\gamma; \frac{g}{z_i}), \quad (2.64)$$

which can itself be transformed into a Stieltjes function:

$${}_2F_0(1, 1+\gamma; \frac{g}{z_i}) = \frac{(-z_i)^{1+\gamma}}{\Gamma(1+\gamma)} \int_0^{+\infty} \frac{e^{z_i z} z^\gamma}{1 + gz} dz = R(g). \quad (2.65)$$

Indeed, as long as  $z_i < 0$  and  $\gamma > -1$ , one obtains  $\rho(z) \propto \frac{(-z_i)^{1+\gamma}}{\Gamma(1+\gamma)} z^\gamma e^{z_i z}$ , which has all the properties it should have for  $R(g)$  to be a Stieltjes function. Thus, for a Stieltjes series (*i.e.*,  $r_k \propto k^\gamma z_i^{-k} k!$  with  $z_i < 0$  and  $\gamma > -1$ ) the normal sequence of Padé approximants converges to the Borel sum. The crucial point is that the Stieltjes integral representation of  $R(g)$  is at most a transformation of variables away from the Borel sum representation. The explicit form of the continued fraction representation for the most general case of a series with coefficients  $r_k = k^\gamma (-1)^k k!$ , valid for all  $\gamma$ 's which are not negative integers, and for  $g$  not on the negative half of the real axis, is [42]:

$$\int_0^{+\infty} \frac{e^{-t}}{(1+gt)^{1+\gamma}} dt = \frac{1}{1 + \frac{(1+\gamma)g}{1 + \frac{g}{1 + \frac{(2+\gamma)g}{1 + \frac{2g}{1 + \dots \frac{(1+\gamma+n)g}{1 + \frac{(n+1)g}{1 + \dots}}}}}}}. \quad (2.66)$$

It must be noted that this remarkable result includes Stieltjes functions as a particular case. In fact, the associated asymptotic series expansion of the integral on the left-hand side of (2.66),

$$\sum_{k=0}^{+\infty} \binom{-(1+\gamma)}{k} \Gamma(k+1) g^k = 1 - (1+\gamma)g + (2+\gamma)(1+\gamma)g^2 - (3+\gamma)(2+\gamma)(1+\gamma)g^3 + \dots, \quad (2.67)$$

is only an alternating-sign Stieltjes series on the special event of  $\gamma > -1$ . Otherwise (say, if  $\gamma \in ]-n_2, -n_1[$  where  $n_1, n_2$  are contiguous integers,  $n_1, n_2 > 0$ ), a finite number of terms will have a fixed sign (constituting a polynomial of  $\mathcal{O}(a^{n_1})$ ), and the alternating-sign pattern will set in from there (*i.e.*, from the  $\mathcal{O}(a^{n_2})$  term). A polynomial can be represented as a continued fraction, and we have seen how any single alternating-sign factorial series is related to a Stieltjes function. Any function of the form (2.66) can thus be decomposed, at worst, as the sum of a Stieltjes series and a polynomial.

Needless to say, all sequences with a  $z_i > 0$  provide examples of non-Stieltjes series which do not converge for  $g$  on the positive real axis. Again, what can be said about linear combinations of alternating-sign factorials? Are they always Stieltjes series? The answer turns out to be *no*, as can be seen in the following example. Given:

$$r_k = ((-1)^k - A(-z_i)^{-k})k!/(1 - A), \quad (2.68)$$

( $z_i > 0$ ) we get

$$\rho(z) = (e^{-z} - Az_i e^{-z_i z})/(1 - A), \quad (2.69)$$

which is non-negative only if  $z_i \geq 1$  ( $A < 1$ ), or  $z_i = 1$  ( $A > 1$ ). Otherwise, the full series is non-Stieltjes. However, there is no reason for the continued fractions of a series which is a linear combination of alternating-sign factorials not to converge to the Borel sum of the same series, even if this is not a Stieltjes function. Indeed, we have numerical evidence that Padé approximants still converge to the Borel sum for non-Stieltjes combinations of alternating-sign factorials. We note in this respect that each individual alternating-sign factorial can be obtained as the asymptotic expansion of a Stieltjes function, even if it is not always possible to write a linear combination of Stieltjes functions as a single Stieltjes function.

### 2.4.1 Hamburger functions

A moment representation including the fixed-sign factorial series can be found by generalising the concept of Stieltjes functions to *Hamburger functions*.

**Definition 5 : Hamburger series** *We have a Hamburger series when there is a real function  $\rho(z) \geq 0 (z \in ]-\infty, +\infty[)$ , so rapidly decreasing when  $z \rightarrow \pm\infty$  that all its moments*

$$r_k = (-1)^k \int_{-\infty}^{+\infty} z^k \rho(z) dz \quad (k = 0, 1, 2, \dots), \quad (2.70)$$

*are finite, and in this case we call the series*

$$\sum_{k=0}^{+\infty} r_k g^k, \quad (2.71)$$

*a Hamburger series.*

As can be easily seen, Stieltjes series are included as a special case, when  $\rho(z) = 0$  for  $z < 0$ . However, the Hamburger equivalent of (2.62),

$$\mathcal{R}(g) = \int_{-\infty}^{+\infty} \frac{\rho(z)}{1 + gz} dz, \quad (2.72)$$

has a cut on the negative real semi-axis. Hamburger functions are thus well-defined for complex values, but convergence cannot be proved in general. As an example, consider the series (2.58), which has the following Hamburger  $\rho(z)$ :

$$\rho(z) = \left( \frac{8}{9} e^{-2z} - \frac{2}{9} e^{-4z} \right) \theta(z) + \left( \frac{16}{9} e^{2z} - \frac{10}{9} e^{4z} \right) \theta(-z) \quad (2.73)$$

(where  $\theta(x)$  is the Heaviside function). Its Stieltjes weight function  $\rho(z)$  would contain exponentials with positive argument.

A result that will be important as a consistency check is that the Padé approximants of a Hamburger series have all their poles on the real axis (Theorem 16.5 of [39]).



### 2.4.2 Stieltjes functions

In Figure 2.3, the resummation of the Stieltjes series (2.31) with  $g = 0.1$  is presented. The normal sequence of Padé approximants converges to the Borel sum (the agreement is of thirteen digits at  $\mathcal{O}(g^{25})$ ), and the power series starts diverging after  $k_{opt.} = 10$  (as expected). It is encouraging to note that even in the lower orders the continued fractions give a better approximation than the power series does. A similar exercise for a fixed-sign factorial series with  $g = 0.1$  would show the Padé approximants never departing radically from the Borel sum, but neither improving its accuracy with higher orders. Also, the distance from the Borel sum at which the values for the fixed-sign factorial continued fractions scatter depends on the size of  $g$ , whereas for the alternating-factorial series continued fractions the size of  $g$  is irrelevant for the convergence properties.

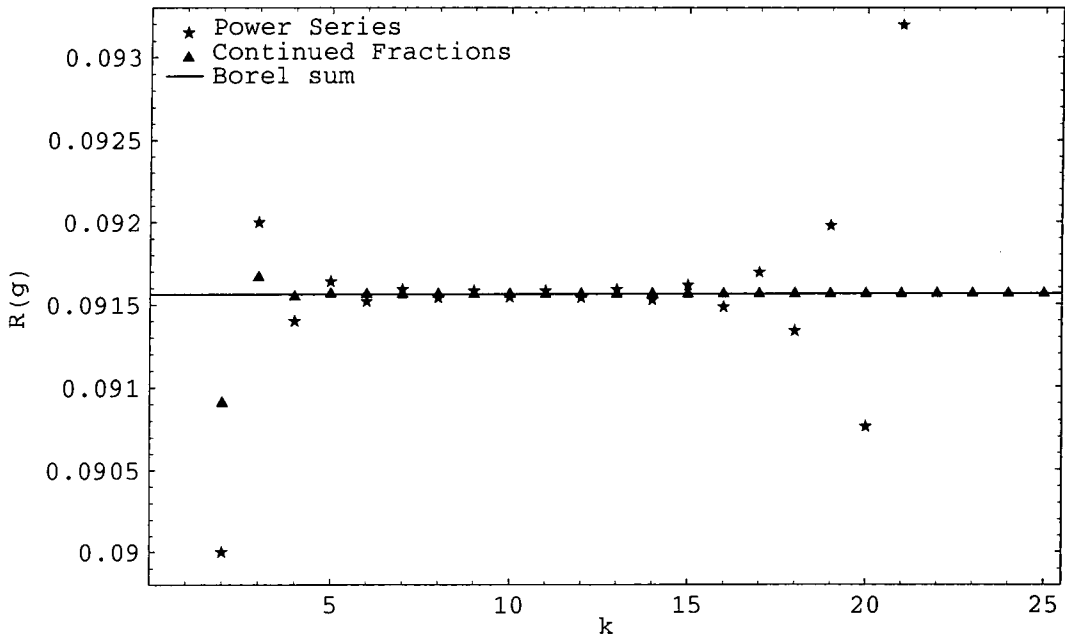


Figure 2.3: Resummation of a Stieltjes series using Padé approximants.

Since Stieltjes series are power expansions of Stieltjes functions, it is of the utmost importance to know if, in general, QCD observables (or, for that matter, QED observables) can be Stieltjes functions. The general properties that a function has to satisfy in order to be a Stieltjes function are the following [40].

**Definition 6 : Stieltjes functions** A function is a Stieltjes function if the following four properties are verified:

- (I)  $R(g)$  is analytic in the cut plane  $|\arg(g)| < \frac{\pi}{2}$ ;
- (II)  $R(g) \rightarrow C$  as  $g \rightarrow +\infty$ , where  $C$  is a real and nonnegative constant;
- (III)  $R(g)$  has a representation in terms of an asymptotic series  $\sum_{k=0}^{+\infty} r_k g^k$  in the cut plane;
- (IV)  $-R(g)$  is Herglotz, i.e.:  $\text{Sign}(\text{Im}(-g)) = \pm \Rightarrow \text{Sign}(\text{Im}(-R(g))) = \pm$ .

The property (I) is assumed without much thought in most physical theories, and certainly in all quantum field theories. However, as we have seen in the first section of this Chapter, it has been proven wrong for the QCD case [29, 34]. The validity of property (II) is not easy to assess. It could only be checked if we had a nonperturbative definition of  $R(g)$ . In fact, perturbative field theories are usually based on the assumption that the coupling is small, and the limit  $g \rightarrow +\infty$  is generally considered unphysical. So nothing can be said about the properties of the full  $R(g)$  in this limit. Property (III) is a general and, as it has been discussed in this Chapter, weak assumption. We can safely assume that it is satisfied. As to property (IV), is not at all clear how to prove or disprove it, but there is no *a priori* reason why it should not be possible to have an Herglotz analytical continuation of  $R(g)$ . So we conclude that a general QCD observable is *not* a Stieltjes function. Reference [29], which assumes only very general properties (momentum-plane analyticity of Green functions and the renormalisation group) is crucial in this conclusion. It is not the fact that there is an essential singularity at the origin which is the problem. It is the zero opening angle which precludes the Stieltjes character and other guarantees of Borel summability <sup>11</sup>. It must however be emphasised that we are assuming that a unique, abstract continuous function  $R(g)$  exists which can represent the observable properties of QCD, and this assumption should be questioned.

As an example of a Stieltjes function, consider the Hamiltonian

$$H_m = p^2 + x^2 + \lambda x^{2m} \quad (\lambda > 0), \quad (2.74)$$

---

<sup>11</sup>The existence of poles on the positive real axis, which will be an important part of the next chapter, is another sufficient reason.

the function representing the energy eigenvalues,  $E_m(\lambda)$ , is a Stieltjes function for  $m=2$  (this corresponds to the quartic anharmonic oscillator) or  $m=3$  [44], and thus one can know exactly the energy levels for these potentials. The same reference acknowledges that the Padé approximants may converge for series which are not Stieltjes series, namely for  $E_m(\lambda)$  with  $m>3$ .

As an another example, the divergent perturbative expansion for the Lagrangian term corresponding to the QED vacuum polarization by an external constant magnetic field  $B$  was shown to be Borel summable in [45], a work which corroborates an earlier result for this semi-classical problem [46]. The exact nonperturbative effective action  $S_B(e)$  for this problem is

$$S_B(g_B) = -\frac{m_e^4 g_B^2}{8\pi^2} \int_0^{+\infty} \left( \coth z - \frac{1}{z} - \frac{z}{3} \right) \frac{e^{-z/\sqrt{g_B}}}{z^2} dz, \quad (2.75)$$

(see, for example, equation (4-123) in [5]), where  $g_B = \frac{e^2 B^2}{m_e^4}$ . Since this result can also be written as [47]

$$S_B(g_B) = -\frac{m_e^4 g_B^2}{4\pi^6} \int_0^{+\infty} \left( \sum_{n=1}^{+\infty} \frac{n^{-4}}{1 + \frac{z}{(n\pi)^2}} \right) z e^{-z/\sqrt{g_B}} dz, \quad (2.76)$$

(a form which is relevant because it shows that all the singularities lie on the negative real axis, specifically at  $z_n = -n^2\pi^2$ ,  $n \neq 0$ ) one then performs a trivial change of variables to arrive at the Stieltjes representation

$$S_B(g_B) = -\frac{m_e^4 g_B^3}{8\pi^2} \int_0^{+\infty} \frac{z}{1 + \sqrt{g_B} z} \left( \vartheta_3(0, e^{-\pi^2 z}) - 1 \right) dz, \quad (2.77)$$

where  $\vartheta_3(u, q)$  is one of the Elliptic Theta functions (see (A.13)).

However, no such exact results are available for any QCD problem, or even for QED. So, one must be resigned to the fact that the guarantees that stem from assuming that QCD observables are Stieltjes functions cannot be applied to generic QCD perturbation series, which are not expected to be Stieltjes. One will have to live with this fact, and learn how to cope with it.

# Chapter 3

## Renormalons

In the previous Chapter, the combinatorial growth in the number of Feynman diagrams at large orders was shown to be a strong enough reason for the coefficients of  $\phi^4$  field theory to be factorially divergent, and thus to give rise to singularities in the Borel transform. Since for simple theories like quantum mechanics and super-renormalisable field theories (one example is  $\phi^4$  theory in three dimensions), the growth of the coefficients themselves is bounded by a power-law, this is expected to be the only source of divergences in these theories, and thus of singularities in the Borel transform (the growth in the number of diagrams is related to instanton singularities). However, in renormalisable theories like  $\phi^4$  in four dimensions, QED, or QCD, there can be divergences arising from certain specific classes of diagrams. In fact, that diagrams containing chains of “bubbles” would make the corresponding field-theoretic amplitudes grow like a factorial of the order was recognised for the first time in the context of two-dimensional asymptotically free field theories [48], and later in QED [49]. In QCD, the character of these additional divergences, which became known as *renormalons*, was first discussed by G. 't Hooft in [34] (the term *renormalon* itself is due to 't Hooft). As will be seen, it turns out that, in QCD, regions of high momentum are related to alternating-sign factorials (*ultraviolet* or *UV* renormalons), and regions of small momentum are related to fixed-sign factorials (*infrared* or *IR* renormalons).

This Chapter is dedicated to the study of renormalons.

### 3.1 The Adler function

In this section we shall see how renormalons arise for one specific physical quantity. This we choose to be the Adler function, which is related to directly measurable observables such as the R-ratio for  $e^+e^-$  annihilation into hadrons, and  $R_\tau$ , the total hadronic width for  $\tau$  decay normalised to the leptonic width.

The correlation function of two vector currents of massless quarks, usually known as *vacuum polarisation function*, is given by

$$i \int e^{iq \cdot x} \langle 0 | T \{ V^\mu(x) V^\nu(0)^\dagger \} | 0 \rangle d^4x = -(g^{\mu\nu} q^2 - q^\mu q^\nu) \Pi(-q^2) \quad (\equiv \Pi_V^{\mu\nu}(q)), \quad (3.1)$$

where  $Q^2 = -q^2$  is the external euclidean momentum, and the vector-isovector current  $V^\mu$  is defined as

$$V^\mu \equiv \frac{1}{2} (: \bar{u} \gamma^\mu u : - : \bar{d} \gamma^\mu d :). \quad (3.2)$$

To avoid an unspecified constant, one defines the Adler function  $\mathcal{A}(Q^2)$  as the logarithmic derivative

$$\mathcal{A}(Q^2) \equiv -Q^2 \frac{\partial \Pi}{\partial Q^2}. \quad (3.3)$$

The perturbation theory expansion of the Adler function can be written as

$$\mathcal{A}(Q^2) = N \sum_f Q_f^2 \left( 1 + \frac{3}{4} C_F D(a) \right) + \left( \sum_f Q_f \right)^2 \tilde{D}(a), \quad (3.4)$$

where  $Q_f$  denotes the electric charge of the quarks (which are summed over the accessible flavours at a given energy), and  $\tilde{D}$  denotes corrections of the “light-by-light” type which first enter at  $\mathcal{O}(a^3)$  and that will be subleading in  $N_f$  (these corrections are expected to be small).  $D(a)$  stands for the QCD perturbation theory corrections to the zeroth order parton model result, it can be expanded as

$$D(a) = a + d_1 a^2 + d_2 a^3 + \dots d_k a^{k+1} + \dots, \quad (3.5)$$

and this is the quantity we shall concentrate on.

The contributions from the vacuum polarisation insertions to the renormalisation of the gluon propagator consist of chains of “bubbles”, such as the ones showed in the

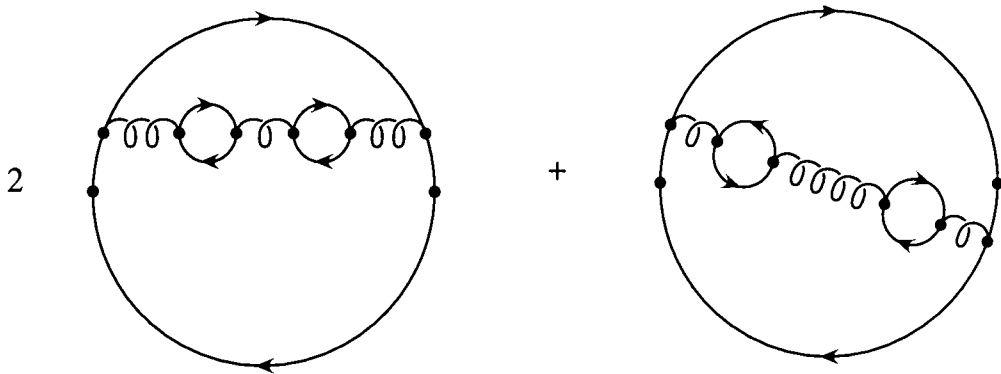


Figure 3.1: The types of diagrams which give rise to renormalons in the Adler function (here shown with two loops inserted).

two types of diagrams in Figure 3.1. The diagrams shown correspond to the leading contributions, since graphs of the type shown in Figure 3.2 would be sub-leading in the coupling for a given number of “bubbles”. Obviously, one has to sum over the contributions from chains with one “bubble”, with two “bubbles”, with three “bubbles”, and so on, and thus one has a series. In QED, there would be only fermionic loops to consider, and these Abelian “bubbles” would contribute a numerical factor proportional to  $N_f$ . In fact, a large- $N_f$  approximation as an organizing principle was implied in the first studies of renormalons [34, 48, 49]. It was not clear at the time what the non-Abelian equivalent should be (and, in rigour, it remains unclear), but it was later suggested that the first  $\beta$ -function coefficient  $b$  could be used for the QCD “bubble”. This was originally termed “leading- $b$  approximation” [50], and it is also referred to as “naive non-Abelianization” [51, 52]. It will be outlined in section 3.2 how this approach can be justified. However, it must be emphasized that there is no direct diagrammatic justification to use the particular combination of gluonic and ghost loops implied in this approximation.

For the time being, let us assume that each isolated QCD “bubble” will correspond to a factor of

$$-\frac{b}{2} \left[ \ln \left( \frac{K_E^2}{\mu^2} \right) + C \right] \quad \left( \equiv \Pi_0(K_E^2) \right), \quad (3.6)$$

where  $K_E^2 \equiv -k^2$  is the euclidean virtual momentum carried by the chain, and  $C$  is a scheme-dependent constant. In the  $\overline{MS}$  scheme,  $C = -\frac{5}{3}$ . We shall use the V-

scheme, which corresponds to the  $\overline{MS}$  scheme with  $\mu = \exp(-5/6)Q$  in the expression above.

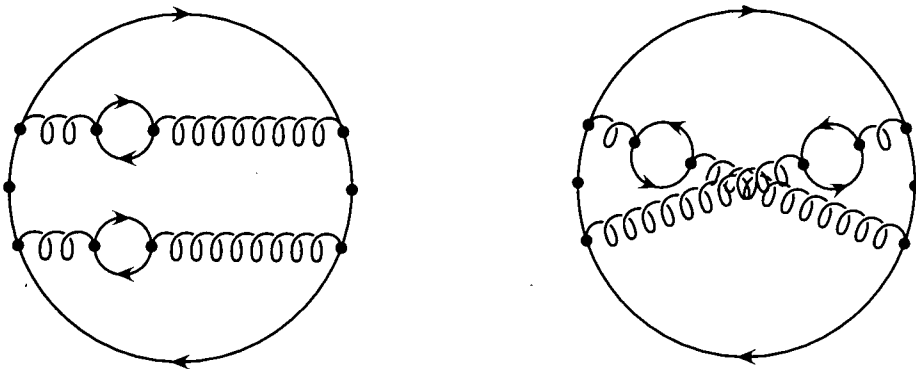


Figure 3.2: The types of diagrams which are sub-leading in the coupling, and thus negligible.

For future reference, we will define now the expansion of each perturbative coefficient in the leading- $N_f$  expansion. So, let us consider the “planar approximation” on which each  $d_k$  is a sum of multinomials in the number of flavours  $N_f$  and in the number of colours  $N$ ,

$$d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} N + d_k^{[k-2]} N_f^{k-2} N^2 + \dots d_k^{[1]} N_f N^{k-1} + d_k^{[0]} N^k, \quad (3.7)$$

such that one has the large- $N_f$  expansion from left to right, and the large- $N$  expansion from right to left. Each coefficient  $d_k^{[k-r]} N^r$  is a sum of multinomials of degree  $r$  on  $C_A$  and  $C_F$ . To arrive at the standard leading- $b$  expansion, one replaces  $N_f \rightarrow \frac{11}{2}N - 3b$  obtaining

$$d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} N + d_k^{(k-2)} b^{k-2} N^2 + \dots d_k^{(1)} b N^{k-1} + d_k^{(0)} N^k, \quad (3.8)$$

where  $d_k^{(k)} = (-3)^k d_k^{[k]}$  exactly. One could equally well consider a “dual  $b$ -expansion” by replacing  $N \rightarrow \frac{6}{11}b + \frac{2}{11}N_f$  in (3.7).

### 3.1.1 The first few Borel plane singularities of the Adler function

The contribution to the Adler function  $D(a)$  from the sum over all the possible, multi-loop, one-chain exchange diagrams such as the ones in Figure 1 is given by

$$\Pi(Q^2) = \sum_{k=0}^{+\infty} a(Q^2) \int_0^{+\infty} \frac{\delta\Pi(Q^2)}{\delta a(K_E^2)} \left[ -\frac{b}{2} a(Q^2) \ln\left(\frac{K_E^2}{\mu^2} e^C\right) \right]^k dK_E^2, \quad (3.9)$$

where  $Q$  is the external momentum, and  $a(Q^2)$  is either the one loop or the two-loop renormalised coupling. The kernel  $\frac{\delta\Pi(Q^2)}{\delta a(K_E^2)}$  corresponds to the forward elastic scattering amplitude of an off-shell vector-isovector quark current (of momentum  $Q$ ) off an off-shell gluon (of momentum  $K_E$ ) evaluated at the one loop level. With  $\hat{k}^2 \equiv \frac{K_E^2}{Q^2}$ , it is given by

$$\frac{\delta\Pi(Q^2)}{\delta a(K_E^2)} = \frac{N}{32\pi^2} \frac{C_F}{Q^2} \begin{cases} \hat{k}^2 \Xi(\hat{k}^2) & \hat{k}^2 \leq 1, \\ \frac{1}{\hat{k}^2} \Xi\left(\frac{1}{\hat{k}^2}\right) & \hat{k}^2 \geq 1, \end{cases} \quad (3.10)$$

where the  $IR \rightleftharpoons UV$  conformal symmetry ( $\hat{k}^2 \leftrightarrow \frac{1}{\hat{k}^2}$ ) is self-evident<sup>1</sup>. This kernel was first mentioned in [54], and, since one only needs to include the appropriate colour factors to use it in QCD, it was recognised in [55] as being of utility in QCD. The function  $\Xi(z)$  itself is defined as

$$\Xi(z) \equiv 1 + \frac{4}{3} \int_0^z \left(1 - \frac{y}{z}\right)^2 \frac{\ln y}{1+y} dy \quad (z \geq 0), \quad (3.11)$$

and the integral above can be explicitly calculated:

$$\Xi(z) = \frac{4}{3} \frac{1}{z} \left\{ 1 - \ln z + \left(\frac{5}{2} - \frac{3}{2} \ln z\right) z + \frac{(1+z)^2}{z} [L_2(-z) + \ln z \ln(1+z)] \right\}, \quad (3.12)$$

where  $L_2(x)$  is the dilogarithm function (A.12). To obtain the Adler function  $D(a)$ , one has to take the logarithmic derivative as defined in (3.3). As a result, the symmetry of the kernel is spoilt, and one has:

<sup>1</sup>The function given here is equivalent to the  $\hat{\omega}_\Pi(\tau)$  of [53]:  $\frac{\delta\Pi(Q^2)}{\delta a(K_E^2)}(IR, UV) = -\frac{N}{96\pi^2} \frac{1}{Q^2} \hat{\omega}_\Pi(\tau, \frac{1}{\tau})$ . The  $F(\hat{k}^2)$  given below needs to be multiplied by  $\frac{N}{24\pi^2} \frac{\hat{k}^2}{Q^2}$  in order to get the  $F(\hat{k}^2)$  of [33].



$$F^{IR}(\hat{k}^2) = \frac{N}{12\pi^2} \frac{C_F}{Q^2} \left\{ \left( \frac{7}{4} - \ln \hat{k}^2 \right) \hat{k}^2 + (1 + \hat{k}^2) [L_2(-\hat{k}^2) + \ln \hat{k}^2 \ln(1 + \hat{k}^2)] \right\}, \quad \hat{k}^2 \leq 1, \quad (3.13)$$

$$F^{UV}(\hat{k}^2) = \frac{N}{12\pi^2} \frac{C_F}{Q^2} \left\{ 1 + \ln \hat{k}^2 + (1 + \hat{k}^2) \left[ L_2\left(-\frac{1}{\hat{k}^2}\right) - \ln \hat{k}^2 \ln\left(1 + \frac{1}{\hat{k}^2}\right) \right] + \left( \frac{3}{4} + \frac{1}{2} \ln \hat{k}^2 \right) \frac{1}{\hat{k}^2} \right\}, \quad \hat{k}^2 \geq 1. \quad (3.14)$$

This function  $F(\hat{k}^2) (\equiv (3/8) \partial(\frac{\delta\Pi}{\delta a}) / \partial \hat{k}^2)$ , considered together with the sum over chains, allows us to obtain an expression for the contribution of renormalons to the Adler function  $D(a)$ :

$$D(a(Q^2)) = \sum_{k=0}^{+\infty} a(Q^2) \int_0^{+\infty} F(\hat{k}^2) \left[ -\frac{b}{2} a(Q^2) \ln\left(\hat{k}^2 \frac{Q^2}{\mu^2} e^C\right) \right]^k d\hat{k}^2. \quad (3.15)$$

This integral is most easily done by expanding  $F^{IR,UV}(\hat{k}^2)$  in a power series:

$$F^{IR}(\hat{k}^2) \propto \frac{3}{4} \hat{k}^2 - \frac{1}{2} \left( \frac{3}{2} - \ln \hat{k}^2 \right) \hat{k}^4 + \frac{1}{6} \left( \frac{5}{6} - \ln \hat{k}^2 \right) \hat{k}^6 + \mathcal{O}(\hat{k}^8), \quad (3.16)$$

$$F^{UV}(\hat{k}^2) \propto \frac{1}{6} \frac{\frac{5}{6} + \ln \hat{k}^2}{\hat{k}^4} - \frac{1}{12} \frac{\frac{7}{12} + \ln \hat{k}^2}{\hat{k}^6} + \frac{1}{20} \frac{\frac{9}{20} + \ln \hat{k}^2}{\hat{k}^8} + \mathcal{O}\left(\frac{1}{\hat{k}^{10}}\right), \quad (3.17)$$

(the overall numerical factor was disregarded here). Obviously, these power expansions are only valid in the small- $\hat{k}$  and large- $\hat{k}$  limits. However, if the renormalisation scale is  $\mu \simeq Q$ , the leading contributions to the integral come indeed from the large logarithmic enhancements at  $K_E \ll Q$  and  $K_E \gg Q$ , and the expressions above can be used in first approximation. We note with respect to the above expansions that the finiteness of the Adler function in both the infrared and the ultraviolet is assured by the power-like structure of  $F(\hat{k}^2)$ . Also,  $F^{IR}(\hat{k}^2)$  and  $F^{UV}(\hat{k}^2)$  have each an infinite number of terms in their power series. We now break the integral (3.15) at  $\hat{k}^2 = \frac{\mu^2}{Q^2} e^{-C}$  into two disjoint parts and perform the two integrations with the expansions (3.16) and (3.17) as the integrands (the necessary generic integrals are given in Appendix D). We obtain a series which consists of a linear combination of alternating-sign and fixed-sign factorial terms:

$$\begin{aligned}
D(a(Q^2)) &= \frac{N}{12\pi^2} \frac{C_F}{Q^2} \sum_{k=0}^{+\infty} \left( \frac{a(Q^2)}{2} \right)^{k+1} \left\{ \frac{3}{4} \left( \frac{b}{2} \right)^k \left( \frac{\mu^2}{Q^2} e^{-C} \right)^2 - \frac{1}{9} \left( k + \frac{11}{2} \right) \left( \frac{b}{3} \right)^k \left( \frac{\mu^2}{Q^2} e^{-C} \right)^3 \right. \\
&\quad \left. + \frac{1}{3} \left( k + \frac{11}{6} \right) (-b)^k \frac{Q^2}{\mu^2} e^C - \frac{1}{24} \left( k + \frac{13}{6} \right) \left( -\frac{b}{2} \right)^k \left( \frac{Q^2}{\mu^2} e^C \right)^2 \right\} k!, \quad (3.18)
\end{aligned}$$

where we considered only the two first terms in both (3.16) and (3.17). The Borel transform can be straightforwardly obtained (we choose the V-scheme at this point):

$$\begin{aligned}
B[D](z) &\propto \frac{3}{4-bz} - \left[ \frac{8}{(6-bz)^2} + \frac{11/3}{6-bz} \right] \\
&\quad + \frac{1}{3} \left[ \frac{8}{(2+bz)^2} + \frac{11/3}{2+bz} \right] - \frac{1}{3} \left[ \frac{4}{(4+bz)^2} + \frac{13/12}{4+bz} \right]. \quad (3.19)
\end{aligned}$$

In the first line of (3.19), we have Borel transform singularities on the positive real axis, which arised from the integration at low momentum, and are henceforth known as *infrared (IR) renormalons*; in the second line of (3.19), we have Borel transform singularities on the negative real axis, which arised from the integration at high momentum, and are henceforth known as *ultraviolet (UV) renormalons*. It is clear that consideration of more terms in the power expansions of  $F(\hat{k}^2)$  would lead to more singularities in the Borel transform, and hence more renormalons. The next ones would be at  $z = \frac{8}{b}$  and  $z = -\frac{6}{b}$ . For the Adler function, the series in (3.16) and (3.17) have no end, and thus an infinite number of  $IR_\ell$  and  $UV_\ell$  renormalons exists, located at  $z_1 = -\frac{2}{b}$  and  $z_\ell = \pm \frac{2\ell}{b}$  ( $\ell \geq 2$ ). The fact that there is no  $IR_1$  renormalon can be traced back to the absence of a constant term in  $F^{IR}(\hat{k}^2)$  (and it is related to the fact that there is no operator of dimension  $1/Q^2$  in the Operator Product Expansion of the Adler function). Furthermore, since, with the notable exception of  $IR_2$ , every ( $IR_\ell$  or  $UV_\ell$ ) renormalon has a structure involving a  $\ln \hat{k}^2$  and a number, multiplied by a (positive or negative) power of  $\hat{k}^2$  (as can be seen in (3.16) and (3.17), respectively), one anticipates a pole + double pole structure for every generic singularity in the Borel sum of the Adler function. However, as it was discussed in sub-section 2.2.1, the simple fact that we have singularities of the Borel sum on the positive side of the axis (and these are the  $IR$  renormalons) precludes a non-ambiguous reconstruction of the observable from the Borel sum. In fact, if we were to assume that a given  $IR_\ell$  were a simple pole, we know from (2.43) that

it would give rise to an ambiguity  $\propto \exp\left(-\frac{z_\ell}{a(Q^2)}\right)$ , which is of the form of a power correction:

$$\delta IR_\ell \approx \left(\frac{\Lambda_{QCD}^2}{Q^2}\right)^\ell, \quad (3.20)$$

provided that we consider the one loop coupling (1.36). Since most  $IR_\ell$  renormalons are double poles, this is true for the leading singularity  $IR_2$  only. If we have a generic double pole, the ambiguity above will get multiplied by a factor of  $\ln \frac{Q^2}{\Lambda_{QCD}^2}$ . How to remove these ambiguities seems to be beyond perturbation theory. It is possible that these ambiguities will be compensated by nonperturbative power corrections [56] associated with non-logarithmic  $UV$  divergences in coefficient functions (see section 3.3).

### 3.1.2 The full singularity structure of the Adler function in the Borel plane

An exact evaluation of the integral (3.15) is needed to obtain the full singularity structure in the borel plane of the Adler function  $D(a)$  large orders. To do this, one may start by noting that  $F^{IR,UV}(\hat{k}^2)$  can be written as a contour integral,

$$F(\hat{k}^2) = \frac{N}{6\pi^3} \frac{C_F}{Q^2} \sum_{k=2}^{+\infty} (-1)^k \frac{d}{dk} \frac{1}{k^2-1} \int_{-\infty}^{+\infty} \exp(ir \ln \hat{k}^2) \left( \frac{1+ir}{r^2+k^2} - \frac{1}{1-ir} \right) dr, \quad (3.21)$$

which separates for values of  $\hat{k}^2 \gtrless 1$ . To check this, one proceeds backwards, by closing the contour in the integral above to separately calculate the residues below the real axis ( $\hat{k}^2 < 1$ ; two poles), and above the real axis ( $\hat{k}^2 > 1$ ; just one pole), and summing the series. We obtain the functions  $F^{IR}(\hat{k}^2)$  and  $F^{UV}(\hat{k}^2)$  in (3.13) and (3.14), respectively. But if, on the other hand, we take the derivative in (3.21), and then we do a trivial change of variables, we arrive at

$$F(\hat{k}^2) = \frac{iN}{32\pi^3} \frac{C_F}{Q^2} \int_{-i\infty}^{+i\infty} P(x) (\hat{k}^2)^{-x} dx, \quad (3.22)$$

where one has defined

$$P(x) \equiv \frac{32}{3(1+x)} \sum_{k=2}^{+\infty} \frac{(-1)^k k}{(k^2 - x^2)^2}. \quad (3.23)$$

Now, if we invert (3.22) for  $P(x)$ ,

$$P(x) = -\frac{16\pi^2}{N} \frac{Q^2}{C_F} \int_0^{+\infty} F(\hat{k}^2) (\hat{k}^2)^{x-1} d\hat{k}^2, \quad (3.24)$$

we see that taking derivatives of  $P(x)$  would bring factors of  $\ln \hat{k}^2$  to the integration above, effectively building the same integrals as in (3.15). Thus,  $P(x)$  can be seen as a generating function for the large orders coefficients of the Adler function  $D(a)$ . This result was obtained for the first time (although following a different path) by D.J. Broadhurst in [57], following progress in applying the  $N_f$  expansion in QED [58]. The actual generating function for the coefficients of the QED Gell-Mann-Low function (MOM scheme  $\beta$ -function) is [57]

$$\Psi_n^{[n]} = \frac{3^{2-n}}{2} \frac{d^{n-2}}{dx^{n-2}} P(x) \Big|_{x=1}. \quad (3.25)$$

This function can be explicitly evaluated in closed form [57]:

$$\begin{aligned} \frac{\Psi_n^{[n]}}{(n-2)!} &= \frac{n-1}{(-3)^{n-1}} \left[ 2(2-n) - \frac{n+4}{2^n} \right. \\ &\quad \left. + \frac{16}{n-1} \sum_{s=1}^{\frac{n}{2}} s(1-2^{-2s})(1-2^{2s-n}) \zeta_{2s+1} \right]. \end{aligned} \quad (3.26)$$

With this result in hand, one can then obtain the leading order large- $N_f$  result for the QCD Adler function. In the  $\overline{MS}$  scheme with  $\mu=Q$ , one has [59]:

$$d_k^{[k]} = 2T_f^k \sum_{m=0}^k \frac{(-\frac{5}{9})^m}{m!} \frac{\Psi_{k+2-m}^{[k+2-m]}}{(k-m)!}, \quad (3.27)$$

where the group theory factor is  $T_f = \frac{1}{2}$  in the standard fermion representation. The  $(-\frac{5}{9})^m$  factor enters since one is converting from the MOM scheme Adler function to that in the  $\overline{MS}$  scheme.

Considerable simplification is achieved by using the V-scheme. The result (3.27) then becomes

$$d_k^{[k]} = 2T_f^k \Psi_{k+2}^{[k+2]}. \quad (3.28)$$

We are now in a position to study the full singularity structure in the Borel plane of the large orders Adler function. Since we are interested in QCD, we have to

convert from the large- $N_f$  expansion  $d_k^{[k]}$ 's into the large- $b$   $d_k^{(k)}$ 's expansion of (3.8). This is simply done replacing  $N_f \rightarrow \frac{11}{2}N - 3b$  (again, why this is reasonable shall be detailed in section 3.2), and one obtains

$$d_k^{(k)} = 2 \left( -\frac{3}{2} \right)^k \Psi_{k+2}^{[k+2]}. \quad (3.29)$$

For the euclidean quantity  $D(a)$  defined in (3.4), it can then be deduced from (3.29) that its Borel transform in the V-scheme is of the form [50]

$$\begin{aligned} B[D](z) = & \sum_{\ell=1}^{+\infty} \frac{A_0(\ell) + A_1(\ell)z + \bar{A}_1(\ell)z + \bar{A}_2(\ell)z^2 + \dots}{\left(1 + \frac{z}{z_\ell}\right)^{\alpha_\ell + \bar{\alpha}_\ell}} \\ & + \sum_{\ell=1}^{+\infty} \frac{B_0(\ell) + B_1(\ell)z + \bar{B}_1(\ell)z + \bar{B}_2(\ell)z^2 + \dots}{\left(1 - \frac{z}{z_\ell}\right)^{\beta_\ell + \bar{\beta}_\ell}} + \dots, \end{aligned} \quad (3.30)$$

where  $z_\ell = \frac{2\ell}{b}$ . The first line corresponds to a summation over the  $UV$  renormalons, and the second line to a summation over the  $IR$  renormalons.  $A_0(\ell)$ ,  $A_1(\ell)$ ,  $\alpha_\ell$ , and  $B_0(\ell)$ ,  $B_1(\ell)$ ,  $\beta_\ell$  can all be obtained from the large- $N_f$  results. The barred terms stand for the unknown sub-leading contributions in  $N_f$ . For a general  $\overline{MS}$  scale,  $\mu = e^{u+v/b}Q$  (where  $u$  is  $N_f$ -independent), an overall factor  $e^{bz(u+5/6+v/b)}$  should multiply the unbarred leading- $N_f$  terms in the numerator. As a matter of fact, the choice of scheme could destroy the dominance of the leading- $b$  term in low orders. For instance, for  $d_k$  in the  $\overline{MS}$  scheme with  $\mu = Q$  ( $u=v=0$ ), and  $d'_k$  with general  $u$  and  $v$ , one has

$$d'_1 = (d_1^{(1)} + u)b + d_1^{(0)} + v, \quad (3.31)$$

$$= d_1 + bu + v, \quad (3.32)$$

and changing  $v$  one could make  $d_1'^{(0)}$  larger than  $d_1'^{(1)}$ , and hence destroy the dominance of the leading- $b$  term. However, the use of the V-scheme guarantees that only the constant and  $\mathcal{O}(z)$  terms in the numerator polynomials are leading in  $N_f$ . It will be seen in sections 3.2 and 3.3 that the  $\bar{\alpha}_\ell$  and  $\bar{\beta}_\ell$  pieces of the exponents, which are sub-leading in  $b$ , are expected to be

$$\bar{\alpha}_\ell = -cz_\ell + \gamma_\ell \quad (3.33)$$

$$\bar{\beta}_\ell = cz_\ell + \gamma'_\ell, \quad (3.34)$$

where  $\gamma_\ell$  and  $\gamma'_\ell$  are the one loop anomalous dimensions of the relevant operators (specifically, see section 3.2.3). However, in the leading- $b$  approximation, these terms are negligible. The exponents of the numerators thus become integers, as already anticipated in (3.19). The coefficients and exponents for the Borel plane singularity structure in (3.30) are then [50]:

$$\begin{aligned} A_0(\ell) &= \frac{8}{3} \frac{(-1)^{\ell+1}(3\ell^2+6\ell+2)}{\ell^2(\ell+1)^2(\ell+2)^2}, & A_1(\ell) &= \frac{8}{3} \frac{b(-1)^{\ell+1}(\ell+\frac{3}{2})}{\ell^2(\ell+1)^2(\ell+2)^2}, & \ell=1, 2, 3, \dots \\ B_0(1) &= 0, & B_0(2) &= 1, & B_0(\ell) &= -A_0(-\ell), \ell \geq 3 \\ B_1(1) &= 0, & B_1(2) &= 0, & B_1(\ell) &= -A_1(-\ell), \ell \geq 3 \\ \alpha_\ell &= 2, \ell=1, 2, 3, \dots; & \beta_2 &= 1, \beta_\ell = 2, \ell \geq 3. \end{aligned} \quad (3.35)$$

The symmetry  $B_{0,1}(\ell) = -A_{0,1}(-\ell)$ , and the additional relation  $A_0(\ell) = -B_0(\ell+2)$  ensure that the constant term in the numerator polynomial for  $UV_\ell$  exactly cancels the one for  $IR_{\ell+2}$ . Therefore,

$$\sum_{\ell=1}^{+\infty} (A_0(\ell) + B_0(\ell)) = B_0(2) = 1, \quad (3.36)$$

which guarantees that the  $\mathcal{O}(a)$  term in the perturbative expansion (3.5) has a unit coefficient. These symmetries are vestiges of the original  $IR \rightleftharpoons UV$  conformal symmetry of the vector correlator in (3.10).

It must be noted that many features of the result (3.30) had already been anticipated in the previous sub-section, namely which singularities existed and their pole + double pole structure. However, since a truncated power expansion never retains the full characteristics of the expanded function, we cannot expect a perfect agreement between the numerical factors in (3.19) and (3.35). We emphasise, though, that the singularity or singularities closest to the origin dominate the asymptotics. In fact, for the Adler  $D(a)$  the leading singularity is  $UV_1$ , and, according to [60], (3.29) becomes

$$d_k^{(k)} \simeq \frac{12k+22}{27} \left(-\frac{1}{2}\right)^k k! \quad \left(=d_k^{(k)}\Big|_{UV_1}\right). \quad (3.37)$$

### 3.1.3 Instantons

Renormalons are not the only singularities in the Borel plane. *Instantons* are solutions of the classical equations of motion which can be related to the factorial growth in the number of Feynman diagrams [30, 34]. In QCD, as in some other theories (for example, the 2-dimensional sigma model of [61]), instanton/anti-instanton pairs ( $I-\bar{I}$ ) produce singularities at  $z=4\ell$ . The resulting structure of singularities in the Borel plane for the Adler function is then as summarised in Figure 3.3.

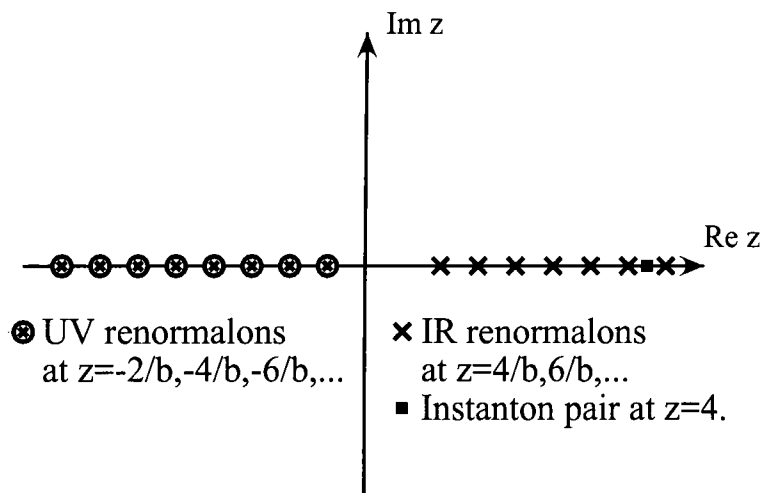


Figure 3.3: Sketch of the Borel plane singularities structure for the Adler function with  $N_f=5$ .

In QED, due to the absence of non-abelian contributions, one has  $b=-\frac{1}{3}N_f$ , with the opposite sign from QCD. Therefore, the *IR* and *UV* renormalons would swap positions if Figure 3.3 were to describe the QED Borel plane structure. However, given the small size of the QED coupling, the ambiguities arising from renormalons are unimportant in QED.

The first  $I-\bar{I}$  pair of the Adler function is a branch point at  $z=4$  with strength  $\gamma=\frac{1}{2}(b-3N_f)$  [62]. The contribution to the asymptotics is found to be of  $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)^{23/3}$ . This is sub-leading as compared to the contributions from any of the first few *IR*

renormalons. That an  $IR$  renormalon will be closer to the origin in the Borel plane rather than the first  $I - \bar{I}$  pair will always be the case for any physically realistic value of the number of flavours, thanks to the fact that the location of instanton singularities is independent of  $N_f$  and  $N$ . Indeed, for  $N_f = 5$ ,  $I - \bar{I}$  is between  $IR_7$  and  $IR_8$ , and one would need  $N_f = 14$  for  $I - \bar{I}$  to become closer to the origin than  $IR_2$  and thus the leading singularity on the right-hand half of the plane.

### 3.1.4 The Adler function in the Hamburger representation

Relating the results of Chapter 2 to the present Chapter, it is immediately obvious that the Borel sum of the Adler function  $D(a)$  can only be defined with some arbitrary prescription to deal with the poles on the positive side of the axis, namely the Cauchy principal value. However, one may wonder if the combination of renormalons in the  $UV$  part of the Adler function  $D(a)$  constitutes a Stieltjes series. To investigate this problem, we start by summing the series and changing variables in (3.15) such that we can write  $D(a)$  in a Hamburger representation (choosing the  $V$ -scheme),

$$D(a(Q^2)) = a(Q^2) \int_{-\infty}^0 \frac{e^s F^{IR}(e^s)}{1 + \frac{b}{2} a(Q^2) s} ds + a(Q^2) \int_0^{+\infty} \frac{e^s F^{UV}(e^s)}{1 + \frac{b}{2} a(Q^2) s} ds. \quad (3.38)$$

The function  $e^s F(e^s)$  is non-negative (see Figure (3.4)) and, as we have seen in the previous sub-section, its moments are all finite (they correspond to the perturbative coefficients in the leading- $b$  approximation). Thus, the second term in the equation above is a Stieltjes function. It must be noted, though, that the exchange in the orders of summation and integration necessary to write (3.38) is forbidden, because we are integrating across the Landau pole in the infrared. This problem has no relation with the existence of  $IR$  renormalons, and neither is it a feature of the one loop coupling. It also arises if we consider the two-loop coupling or the renormalised coupling in any other finite renormalisation scheme. The fact that  $D^{IR}(a(Q^2))$  has a singularity along the domain of integration reminds us that the long-distance effects of QCD are not fully contained in the perturbative information.

The fact that the weight function  $\rho(s) = e^s F(e^s)$  is the result of the all-orders summation of diagrams integrated over all scales of the running coupling in the



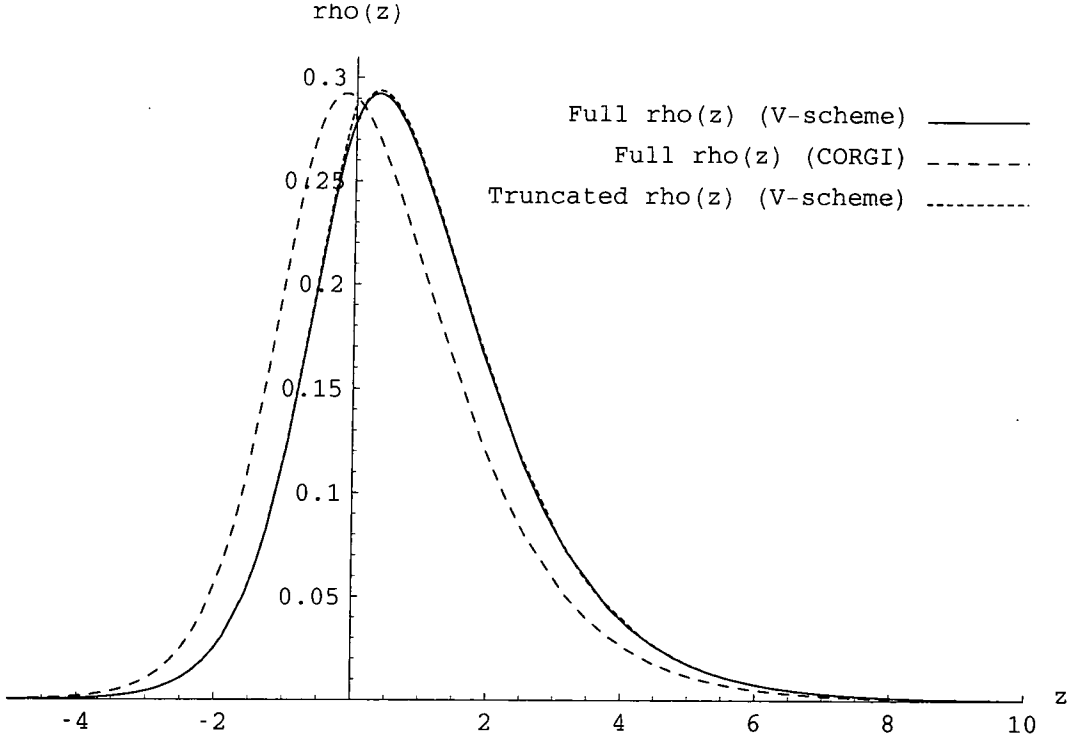


Figure 3.4: The Hamburger weight function  $\rho(z)$  in the V-scheme, in the CORGI scheme, and in the V-scheme calculated with the series in equations (3.41-3.42) truncated at  $\ell=8$ .

leading- $b$  limit, helps clarifying its physical meaning. In fact, the weight function can be understood as the distribution function of the momentum flowing through the gluon line [53, 63].

One can also investigate if  $D^{UV}(a(Q^2))$  is a Stieltjes function by writing the Borel sum of the Adler function  $D(a)$  making use of (3.30) and of the coefficients given in (3.35). Neglecting the sub-leading terms in  $b$  ( $\bar{A}_1(\ell)$  and  $\bar{A}_2(\ell)$ ), one has

$$\begin{aligned}
 D(a) = & \int_0^{+\infty} e^{-z/a} e^{bz(u+5/6+v/b)} \sum_{\ell=1}^{+\infty} \frac{A_0(\ell) + A_1(\ell)z}{\left(1 + \frac{z}{z_\ell}\right)^\gamma} dz \\
 & + \int_0^{+\infty} e^{-z/a} e^{bz(u+5/6+v/b)} \left( \frac{B_0(2)}{1 - \frac{z}{z_2}} + \sum_{\ell=3}^{+\infty} \frac{B_0(\ell) + B_1(\ell)z}{\left(1 - \frac{z}{z_\ell}\right)^\gamma} \right) dz, \quad (3.39)
 \end{aligned}$$

where we assume  $\overline{MS}$  subtraction with  $\mu = e^{u+v/b}Q$ , and  $\gamma$  is introduced as the

general exponent of the Borel plane singularities only to keep the results in the following as general as possible. Now, if we choose the V-scheme ( $u = -\frac{5}{6}$  and  $v = 0$ ), the integral in the first line of (3.39) can be transformed, via relation (2.40) and a change of variables, into the Stieltjes-like form

$$D^{UV}(a) = a \int_0^{+\infty} \frac{\rho_0(t) + \rho_1(t)}{1 + \frac{b}{2}at} dt, \quad (3.40)$$

where

$$\rho_0(t) = \sum_{\ell=1}^{+\infty} \frac{A_0(\ell)\ell}{\Gamma(\gamma)} (\ell t)^{\gamma-1} e^{-\ell t}, \quad (3.41)$$

$$\rho_1(t) = \sum_{\ell=1}^{+\infty} \frac{A_1(\ell)z_\ell\ell}{\Gamma(\gamma)} (\ell t)^{\gamma-1} e^{-\ell t} \left[ \frac{\gamma-1}{\ell t} - 1 \right]. \quad (3.42)$$

It must be noted that the weight functions  $\rho_0(t)$  and  $\rho_1(t)$  are completely independent from  $b$ . More to the point, the function  $\rho_0(t)$ , by itself, is non-negative and goes to zero at infinity, driven by the exponentials  $e^{-\ell t}$ . The combination  $\rho_0(t) + \rho_1(t)$  shares the same properties<sup>2</sup>. Thus, we have seen again that  $D^{UV}(a)$  in the V-scheme is a Stieltjes function<sup>3</sup> (see also Figure 3.4).

As a consequence of the arguments put forward in this sub-section, if we were to consider only the  $UV$  renormalons, at least the normal sequence of Padé approximants to the series  $D^{UV}(a)$  would converge to the Borel sum, provided the chosen RS was the V-scheme, and this would be guaranteed by Theorem 4. Since  $IR$  renormalons are unavoidably present in the full large-orders structure of the Adler function  $D(a)$ , we do not have such guarantees. Anyhow, it will be instructive to study how the predominantly Stieltjes combination of renormalons in the Adler function  $D(a)$  will compensate (or not) for the  $IR$  renormalons when resummed with Padé approximants.

It must be emphasised that these considerations are RS-dependent, and that by varying the RS one can destroy the Stieltjes character of  $D^{UV}(a)$ . As a proof, let

<sup>2</sup>The same exercise for the  $IR$  part (second line of (3.39)) would lead to a  $\rho^{IR}(t)$  dominated by exponentials with positive argument (e.g.,  $e^{\ell t}$  in the V-scheme).

<sup>3</sup>We have taken the realistic  $\gamma = 2$  for the numerical experiments, but any  $\gamma \geq 1$  would lead to the same conclusion. The first three  $UV$  renormalons are actually enough to give more than 99% of the numerical content of these functions.

us consider a given scheme RS on which we have a Stieltjes representation for a quantity  $R(a)$ , and let us transform this Stieltjes function into a new scheme RS' via the NLO RS-invariant  $\rho_0, \frac{1}{a} - r_1^{RS} = \frac{1}{a'} - r_1^{RS'}$ . After a change of variables, the relation between the Stieltjes-like functions in the two schemes is

$$a \int_0^{+\infty} \frac{\rho(z)}{1+az} dz = a' \int_0^{+\infty} \theta(z - (r_1^{RS} - r_1^{RS'})) \frac{\rho(z - (r_1^{RS} - r_1^{RS'}))}{1+a'z} dz \quad (3.43)$$

(where  $\theta(z)$  is the Heaviside step function). We see that if  $r_1^{RS} > r_1^{RS'}$ ,  $R(a')$  is Stieltjes, but otherwise it is not. As a matter of fact, we have seen how we can obtain a Stieltjes representation for the Adler function  $D^{UV}(a)$  in the V-scheme, and we can now realise that in the CORGI scheme where  $(d_1^{EC} = 0) > (d_1^V < 0)^4$ ,  $D^{UV}(a)$  is *not* a Stieltjes function. This is due to the fact that the shift in the argument of the Stieltjes representation shifts part of the Stieltjes weight function into the negative side of the real axis, and the actual coefficients are then generated with a contribution from this part, which amounts to a (sub-leading) fixed sign factorial. On the other hand, if a given quantity has a Hamburger representation, a change of renormalisation scheme does not destroy its Hamburger character, because the domain where the moments are calculated extends to the whole real axis.

## 3.2 Motivating the leading- $b$ approximation

In this section, we shall be concerned with the motivation of the leading- $b$  approximation.

As a matter of fact, it was assumed in the previous section, as it is always implicitly assumed in renormalon calculus, that an effective charge exists in QCD analogous to the one of QED. It is indeed possible to define an effective charge in QED by summing the infinite subset of one-particle-irreducible vacuum polarisation insertions which occur when renormalising the photon propagator. This defines an effective charge which is gauge-independent, scheme-independent, and matches on to the renormalisation group running coupling in the  $k^2 \rightarrow \infty$  limit, and to the bare charge (the fine structure constant) in the  $k^2 \rightarrow 0$  limit:

---

<sup>4</sup>It must be noted that both  $d_1^V(UV+IR)$  and  $d_1^V(UV)$  are negative.

$$\bar{a}(k^2) = \frac{e^2}{4\pi} \frac{1}{1 + \Pi(k^2)}, \quad (3.44)$$

where  $e$  and  $\Pi(k^2)$  are bare quantities,  $\Pi(k^2)$  being simply one of the fermionic loops.

In QCD, the sum of all fermionic loops insertions is not the only contribution to the gluon self-energy, since there are also contributions from non-Abelian loops which have to be considered. Nevertheless, it is possible to sum the series in the renormalisation of the gluon propagator to get a corrected gluon propagator, and this corresponds to replacing the ordinary bare gluon propagator by an effective propagator. In a covariant gauge, this amounts to

$$\frac{i}{k^2} \left( -g_{\mu\nu} + (1-\zeta) \frac{k_\mu k_\nu}{k^2} \right) \rightarrow \frac{i}{k^2} \left( -g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{1 + \Pi_0(k^2)} - i\zeta \frac{k_\mu k_\nu}{k^4} \quad (3.45)$$

(see (3.6) for the definition of  $\Pi_0(k^2)$ ), but the quantity defined by analogy with QED is gauge-dependent due to consideration of the gluon and ghost loops, and scheme-dependent because there is no low energy fixed point in QCD to match the running coupling to. Thus, there is no correspondence between the gluon self-energy and the effective charge of QCD.

Using the background field method (BFM), the one loop gluon vacuum polarization diagram has been evaluated to be [64]

$$\hat{\Pi}_{\overline{MS}}(K_E^2, \zeta_Q) = a_{\overline{MS}}(\mu) \left\{ -\frac{b}{2} \left( \ln \frac{K_E^2}{\mu^2} - \frac{5}{3} \right) + \frac{N}{3} \left[ 1 - \frac{3}{16} (1 - \zeta_Q)(7 + \zeta_Q) \right] \right\}, \quad (3.46)$$

where  $\overline{MS}$  has been used and  $\zeta_Q$  is the background field gauge parameter. In the background field method, one separates the gauge fields into background and quantum components. The gauge-fixing for the quantum fields is made such that the effective action remains invariant under gauge transformations of the background fields. Invariance with respect to gauge transformations of the background fields does not imply independence with respect to the changes in the quantum gauge parameter. From (3.46), implementing the leading  $b$  approximation can now be recognised as being equivalent to neglecting the term in (3.46) which is not proportional to  $b$ , and which is dependent on the BFM gauge. The BFM gauge dependence

can also be absorbed into the scale  $\mu$ , a fact which follows from Kallosh's Theorem [65]. This has the useful consequence that scheme invariant combinations such as  $d_2^{(2)} - d_1^{(1)2}$  are independent of  $\zeta_q$ . We shall base our all-orders resummations on RS-invariants  $X_k^{(L)}$  (see sub-section 5.1.1). Possible choices for  $\zeta_Q$  are  $\zeta_Q = 0$  (Landau BFG),  $\zeta_Q = -3$  (minimal non-logarithmic contribution in (3.46)), and  $\zeta_Q = 1$ , which coincides with the expression from the pinch technique developed in [66, 67]. In fact, the pinch technique has been valued as a way to identify a QCD effective charge [68]. At the one loop level, it is related to

$$\hat{\Pi}_{\overline{MS}}(K_E^2) = a_{\overline{MS}}(\mu) \left\{ -\frac{b}{2} \left( \ln \frac{K_E^2}{\mu^2} - \frac{5}{3} \right) + \frac{1}{3} N \right\}, \quad (3.47)$$

which is gauge-independent. Work is still in progress on the application of the pinch technique beyond one loop [69]. It is important to note with respect to the the background field method that, by construction, the background fields do not propagate inside loops. Thus, the BFM effective charge cannot be used to justify the use of the leading- $b$  for a quantum gluon inside, for example, a fermion loop.

The leading- $b$  approximation can be implemented directly on the QED results. This amounts to replace  $N_f \rightarrow \frac{11}{2}N - 3b$ , and it was the approach taken from equation (3.28) onwards. One can also assume from the beginning that a QCD “bubble” corresponds to a factor proportional to the first  $\beta$ -function coefficient  $b$  (calculated in sub-section 1.3.1), and this was the approach taken in sub-section 3.1. Either way, one is assuming that the particular combination of gluons and ghosts loops which is responsible for the one loop running of the coupling plays a special role in all orders of perturbation theory. However, there is no direct diagrammatic justification for this assumption. In the following, we shall advance several arguments in order to justify that the leading- $b$  approximation is nevertheless plausible.

### 3.2.1 The large $b$ limit and the renormalised coupling

The skeleton expansion discussed in [70] can shed some light on what one is doing when using the leading- $b$  approximation. Let us assume that a generic QED perturbative series can be written in the form of a skeleton expansion

$$R(Q^2) = R_0(Q^2) + s_1 R_1(Q^2) + \dots s_k R_k(Q^2) + \dots, \quad (3.48)$$

where  $R_0$  is obtained by including all possible vacuum polarisation insertions into a single photon line,  $s_1 R_1$  corresponds to all possible insertions into a double-photon exchange, and so on. Since we have been interested in the vacuum polarisation function, we shall thus concentrate on  $R_0$ . In QED,  $R_0$  can be written as

$$R_0(Q^2) \equiv \int_0^{+\infty} \bar{a}(K_E^2) F(\hat{k}^2) dK_E^2, \quad (3.49)$$

where  $K_E^2$  is the virtual momentum of the exchanged photon,  $\bar{a}(K_E^2)$  is the renormalised effective charge representing the full propagator, and  $F(\hat{k}^2)$  can be interpreted, again, as the photon momentum distribution function [53].

In QCD, the equivalent of  $R_0$  is no other than the Adler function  $D(a)$ . Then, since at the physical scale  $\mu = Q$  the QCD renormalised coupling can be expanded as

$$\begin{aligned} a(K_E^2) &= a(Q^2) - b \ln \frac{K_E}{Q} a^2(Q^2) + \left( b^2 \ln^2 \frac{K_E}{Q} - cb \ln \frac{K_E}{Q} \right) a^3(Q^2) \\ &\quad - \left( b^3 \ln^3 \frac{K_E}{Q} - \frac{5}{2} cb^2 \ln^2 \frac{K_E}{Q} + c_2^{RS} b \ln \frac{K_E}{Q} \right) a^4(Q^2) + \dots, \end{aligned} \quad (3.50)$$

if we insert this expansion under the integral sign in (3.49), we obtain

$$R_0(Q^2) = a(Q^2) + d_1^{(1)} a^2(Q^2) + \left( d_2^{(2)} + c d_1^{(1)} \right) a^3(Q^2) + \left( d_3^{(3)} + \frac{5}{2} c d_2^{(2)} + c_2^{RS} d_1^{(1)} \right) a^4(Q^2) + \dots, \quad (3.51)$$

where

$$d_k^{(k)} \equiv \int_0^{+\infty} \left[ -\frac{b}{2} \ln \frac{K_E^2}{Q^2} \right]^k F(\hat{k}^2) dK_E^2. \quad (3.52)$$

Thus one notes that taking the large  $b$  limit in (3.50) (that is, ignoring terms which are sub-leading in  $b$ ), leads to  $d_k = d_k^{(k)}$  in (3.51). This shows that the choice of (3.6) as the leading term in the renormalisation of the gluon propagator is clearly consistent with retaining only the leading logarithms in the renormalisation of the coupling.

### 3.2.2 Comparison between leading- $b$ estimates and exact coefficients

Even if the leading- $b$  approximation is believed to work better at large orders, one can check its utility for the low orders perturbative coefficients for which numerical results are available. So, let us consider the expansion in  $N_f$  (3.7), and the expansion in  $b$  (3.8). The “dual  $b$ -expansion” is of little practical interest since what we have available are all-orders large- $N_f$  results. The standard “ $b$ -expansion” is exact in the large- $N_f$  limit (as the “dual  $b$ -expansion” is exact in the large- $N$  limit), but it only provides a estimate of the coefficients which are sub-leading in  $N_f$ :

$$d_k^{(k)} b^k = \tilde{d}_k^{[k]} N_f^k + \tilde{d}_k^{[k-1]} N_f^{k-1} N + \tilde{d}_k^{[k-2]} N_f^{k-2} N^2 + \dots \tilde{d}_k^{[1]} N_f N^{k-1} + \tilde{d}_k^{[0]} N^k. \quad (3.53)$$

As a matter of fact, by making use of the operator analysis of [71], one can show that [72]

$$d_k^{[k-r]} \simeq \tilde{d}_k^{[k-r]} \left[ 1 + \mathcal{O}\left(\frac{1}{k}\right) \right] \quad (r < k), \quad (3.54)$$

whereas  $d_k^{[k]} = \tilde{d}_k^{[k]}$  exactly by construction. Thus, the leading- $b$  coefficients provide a estimate of the terms which are sub-leading in  $N_f$ . Since we have at our disposal exact results for  $d_1$  and  $d_2$ , one can assess the accuracy of those estimates by comparing them with the exact numbers. Using  $d_1$  and  $d_2$  of the Adler function  $D(a)$  in the  $\overline{MS}$  with  $\mu = Q$  [73, 74], the leading- $b$  terms hold

$$d_1^{(1)} b = -0.115 N_f + 0.634 N, \quad (3.55)$$

$$d_2^{(2)} b^2 = 0.086 N_f^2 - 0.948 N_f N + 2.61 N^2, \quad (3.56)$$

while exactly,

$$d_1 = -0.115 N_f + \left( 0.655 N + \frac{0.063}{N} \right), \quad (3.57)$$

$$d_2 = 0.086 N_f^2 + \left( -1.40 N - \frac{0.024}{N} \right) N_f + \left( 2.10 N^2 - 0.661 - \frac{0.180}{N^2} \right). \quad (3.58)$$

So we see that the leading- $b$  coefficients seem to approximate rather well the sub-leading terms in  $N_f$  in both sign and magnitude for the Adler function  $D(a)$  lowest

orders perturbative coefficients. Obviously, the full coefficients  $d_k$  and  $d_k^{(k)}b^k$  disagree. For example,  $d_2^{(2)}b^2=15.7$ , whereas  $d_2=6.37$  with  $N_f=3$ . The same exercise for the polarised Bjorken or GLS sum rule holds similarly encouraging results, as it is also the case for the unpolarised Bjorken sum rule [75]. For a leading- $b$  approximation on the  $\beta$ -function RS-invariants  $\rho_k$ , the comparison is also encouraging for the Adler function and for the polarised Bjorken or GLS sum rule [75]. Nevertheless, the “dual  $b$ -expansion” seems to work better than the standard leading- $b$  approximation for the perturbative coefficients of the Higgs decay width  $\Gamma(H \rightarrow b\bar{b})$  [76].

### 3.2.3 The leading- $b$ approximation in operator analysis

That the large orders coefficients in QCD perturbation theory are expected to be a linear combination of terms of the form  $k^\gamma z_\ell^{-k} k! \left[1 + \mathcal{O}(1/k)\right]$  can be understood by studying the insertions of operators into the Green functions that allow us to calculate the observables. In this sub-section we shall be interested in the operators that give rise to the large-orders behaviour of QCD in the ultraviolet limit. As was seen before, these operators correspond to singularities on the negative side of the real axis. So, let us assume with G. Parisi [77] that each of these singularities (which one will assume that correspond to an imaginary part of the Borel sum) can be factorised as

$$\delta R(a) = \frac{1}{\mu^2} \sum_i C_i(a) R_{\mathcal{O}_i}(a), \quad (3.59)$$

where  $C_i(a)$  are the observable-independent coefficient functions, and  $R_{\mathcal{O}_i}$  is the Green function from which  $R(a)$  is derived with a single zero-momentum insertion of the dimension-six operators  $\mathcal{O}_i$ . These operators can be thought of as an additional term in the QCD Lagrangian,

$$\Delta\mathcal{L} = -\frac{i}{\mu^2} \sum_i C_i(a) \mathcal{O}_i, \quad (3.60)$$

with coefficients  $C_i(a)$  chosen to compensate for the imaginary part of the Borel sum. From the RG equations for these operators (see sub-section 3.3.1 for more detail in a similar calculation), one derives [33] that



$$C_i(a) = \exp\left(-\frac{1}{ba}\right) a^{-c/b} F(a) E_i(a), \quad (3.61)$$

where

$$F(a) = \exp\left\{\int_0^{a(Q^2)} \left[\frac{1}{x^2(1+cx)} - \frac{1}{x^2 B(x)}\right] dx\right\} \quad (3.62)$$

takes care of the higher-order RG effects, and

$$E_i(a) = \tilde{C}_i \exp\left\{\int_{a(Q_0^2)}^{a(Q^2)} \frac{\gamma_{ij}(x)}{2x^2 B(x)} dx\right\}, \quad (3.63)$$

takes into account the anomalous dimension matrix  $\gamma_{ij}(x)$  (the  $\tilde{C}_i$  are integration constants which depend on the arbitrary limit of integration  $a(Q_0^2)$ ).

To proceed, one has to specify a basis of dimension-six operators. For simplicity, only vector and axial-vector currents will be considered. Since one is also interested in external currents, two  $U(1)$  background fields  $v_\mu$  and  $a_\mu$  are introduced. These have field strengths  $U_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$  and  $H_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ , and couple to the vector and axial-vector currents, respectively. The basis is then:

$$\mathcal{O}_1 = (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi), \quad (3.64)$$

$$\mathcal{O}_2 = (\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma^\mu\gamma_5\psi), \quad (3.65)$$

$$\mathcal{O}_3 = (\bar{\psi}\gamma_\mu T^a\psi)(\bar{\psi}\gamma^\mu T^a\psi), \quad (3.66)$$

$$\mathcal{O}_4 = (\bar{\psi}\gamma_\mu\gamma_5 T^a\psi)(\bar{\psi}\gamma^\mu\gamma_5 T^a\psi), \quad (3.67)$$

$$\mathcal{O}_5 = \frac{1}{g_s} f_{abc} F_{\mu\nu}^a F_\rho^{\nu,b} F^{\rho\mu,c}, \quad (3.68)$$

$$\mathcal{O}_6 = \frac{1}{g_s^2} (\bar{\psi}\gamma_\mu\psi) \partial_\nu U^{\nu\mu}, \quad (3.69)$$

$$\mathcal{O}_7 = \frac{1}{g_s^2} (\bar{\psi}\gamma_\mu\gamma_5\psi) \partial_\nu H^{\nu\mu}, \quad (3.70)$$

$$\mathcal{O}_8 = \frac{1}{g_s^4} \partial_\nu U^{\nu\mu} \partial^\rho U_{\rho\mu}, \quad (3.71)$$

$$\mathcal{O}_9 = \frac{1}{g_s^4} \partial_\nu H^{\nu\mu} \partial^\rho H_{\rho\mu}, \quad (3.72)$$

where a sum over flavour, colour and spinor indices is implied in each term of the form  $(\bar{\psi} M \psi)$ , and  $M$  does not act in flavour space. Quarks are considered massless.

The leading-order anomalous dimension matrix can then be determined [33]. One obtains

$$E_i(a) = \sum_{j=1}^4 C_{ij}^{[1]} a^{-\lambda_j} \quad i = 1, 2, 3, 4 \quad (3.73)$$

$$E_5(a) = C_5^{[1]} a^{-\lambda_5} \quad (3.74)$$

$$E_i(a) = C_i^{[2]} a + \sum_{j=1}^4 C_{ij}^{[1]} a^{-\lambda_j} \quad i = 6, 7 \quad (3.75)$$

$$E_i(a) = C_i^{[2]} a + C_i^{[3]} a^2 + \sum_{j=1}^4 C_{ij}^{[1]} a^{-\lambda_j} \quad i = 8, 9 \quad (3.76)$$

The exponents  $\lambda_j$  and the coefficients  $C_i$  are given in [33]. Also, at leading order,  $F(a)=1$ . Furthermore [33],

$$R_{\mathcal{O}_i}(a) \propto a^0 \quad i = 1, 2, 3, 4 \quad (3.77)$$

$$R_{\mathcal{O}_5}(a) \propto a \quad (3.78)$$

$$R_{\mathcal{O}_i}(a) \propto a^{-1} \quad i = 6, 7 \quad (3.79)$$

$$R_{\mathcal{O}_i}(a) \propto a^{-2} \quad i = 8, 9 \quad (3.80)$$

Overall constants were ignored. Now, it follows trivially comparing (3.59) with (2.43) that

$$d_k = \left[ \sum_{i=1}^4 A_i k^{2+\lambda_i} + A_5 k^{-1+\lambda_5} + A_6 + A_8 k \right] k^{c/b} b^k k! \left[ 1 + \mathcal{O}\left(\frac{1}{k}\right) \right]. \quad (3.81)$$

The operators  $\mathcal{O}_{7,9}$  were discarded because we concentrate on vector currents. The normalisation constants  $A_i$  remain undetermined, a fact which reminds us that we do not know exactly the coefficients at large orders. However, the position of the Borel sum singularities is clearly at  $z \sim 1/b$ , following the effective leading- $b$  approximation implied in taking  $F(a) = 1$ , and the strength of the singularities follows from the operators relevant for each observable, plus an universal  $c/b$  term.

The leading asymptotic behaviour for the Adler  $D(a)$  according to (3.81) is then

$$d_k = A k^{\{1.59, 1.75, 1.97\}} b^k k!, \quad (3.82)$$

for  $N_f = \{3, 4, 5\}$ . This must be compared with (3.37).

We can now see that the leading- $b$  approximation, also in operator analysis, is consistent with the position of the singularities in the Borel plane being at  $1/b$ .

### 3.3 Power corrections

A question remains open from section 3.1. Since all QCD observables will have  $IR$  renormalons, all the resummation methods studied in Chapter 2 are bound to have problems. This implies that perturbative QCD is not well-defined by itself, and thus that some extra, nonperturbative component is needed to render the theory meaningful. Furthermore, that QCD is incomplete and that power corrections are needed is already clear if we consider that since the running coupling ought to be analytical, therefore we should remove the Landau pole from the one loop perturbative coupling:

$$\hat{a}(Q^2) \equiv \frac{1}{\frac{b}{2} \ln\left(\frac{Q^2}{\Lambda_{QCD}^2}\right)} + \frac{\Lambda_{QCD}^2}{\frac{b}{2}(\Lambda_{QCD}^2 - Q^2)}. \quad (3.83)$$

The coupling  $\hat{a}(Q^2)$  defined above has no Landau pole (just a branch point at  $Q^2 = 0$ ), but at large  $Q^2$  it differs from  $a^{1-loop}(Q^2)$  by a  $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$  power correction. This is akin to the “analytic perturbation theory” approach [16]. The existence of the Landau pole is unrelated to the existence of  $IR$  renormalons, but stresses that truncated perturbative QCD has to be supplemented by power corrections. More generally, a generic QCD observable  $R(Q^2)$  is not fully described by its perturbative expansion, even if known to all orders, and the nonperturbative terms can be parameterised as power corrections:

$$R(Q^2) = R(Q^2)_{parton\ model} \left( \sum_{k=0}^{+\infty} d_k a^{k+1}(Q^2) + \sum_{j=N}^{+\infty} B_j \left( \frac{\Lambda_{QCD}^2}{Q^2} \right)^j \right). \quad (3.84)$$

These power corrections may play an important role in the long-distance region where perturbative QCD fails. Concerning perturbative QCD, we already saw how  $IR$  renormalons indicate the existence and form of power corrections. However, the converse is not true. There may be power corrections not hinted at by renormalons, especially in time-like processes. Besides renormalons, the form of power corrections

can also be suggested by the Operator Product Expansion (OPE), even if, again, there are observables which do not admit an OPE but have power corrections. We shall describe now how the OPE can be used.

### 3.3.1 The Operator Product Expansion

It was first suggested in [78] that two local operators  $A$  and  $B$  could have a short-distance expansion such as

$$A(x)B(y) \sim \sum_i C_i(x-y) \mathcal{O}_i\left(\frac{x+y}{2}\right) \quad (x \rightarrow y), \quad (3.85)$$

where the  $\mathcal{O}_i$  are a sequence of local regular operators, while the  $C_i(x-y)$  are c-numbers which are singular on the limit  $x \rightarrow y$ . The  $\mathcal{O}_i$ 's and  $C_i$ 's are related; the higher the dimension of a given  $\mathcal{O}_i$ , the faster the respective  $C_i$  has to go to zero:

$$\lim_{x \rightarrow 0} C_i(x) \sim x^{\gamma_i} \quad (\text{mod } \ln|x|), \quad (3.86)$$

where the dimension  $\gamma_i$  is typical of the operators considered ( $\gamma_i = d_{\mathcal{O}_i} - d_A - d_B$ ),  $d_{\mathcal{O}_i}, d_A, d_B$  being the anomalous dimensions of the operators  $\mathcal{O}_i, A, B$ .

One starts by separating our observable  $R(Q^2)$  into a perturbative part, and a nonperturbative (condensates) part:

$$R(Q^2) = R^{PT}(a) + R^{OPE}(a). \quad (3.87)$$

In our case, one is interested in the OPE of the vacuum polarisation function defined in (3.1). Its calculation was carried out in [79, 80]. One knows from the renormalon analysis that a condensate of dimension minus four in the external scale  $Q$  must exist (to compensate for  $IR_2$ ). The corresponding parameter must be the matrix element of a local operator. This operator must be bilinear in the gluon fields, since one is renormalising a single gluon line. Since the Adler function is a Lorentz scalar, and  $G_\mu G^\mu$  is not gauge invariant, we have to consider covariant derivatives acting on the product of two field strength tensors with all the Lorentz indices contracted. One is then uniquely led to introduce the *gluon condensate* [80] of lowest dimension (four)

$$G_0(a) = \frac{\langle 0 | F_{\mu\nu}^a F^{a,\mu\nu} | 0 \rangle}{Q^4}(\mu) C_{GG}\left(\frac{Q^2}{\mu^2}\right), \quad (3.88)$$

where  $C_{GG}\left(\frac{Q^2}{\mu^2}\right)$  is the Wilson coefficient for the gluon condensate in the OPE. Of course, the full Operator Product Expansion of the current correlation function at this level is

$$R^{OPE}(Q^2) = \frac{1}{Q^4} \left[ \langle 0 | F_{\mu\nu}^a F^{a,\mu\nu} | 0 \rangle(\mu) C_{GG}\left(\frac{Q^2}{\mu^2}\right) + m_q \langle 0 | q\bar{q} | 0 \rangle(\mu) C_{q\bar{q}}\left(\frac{Q^2}{\mu^2}\right) \right] + \mathcal{O}\left(\frac{1}{Q^6}\right), \quad (3.89)$$

but the quark condensate is obviously absent on the limit of massless quarks. Now, since  $R^{PT}(Q^2)$  and  $R^{OPE}(Q^2)$  must be separately RG invariant, we know from renormalisation group arguments that we expect the condensate  $G_0(a)$  of dimension  $d$  to be given by

$$G_0(a(Q^2)) = C(a(Q^2)) \exp \left[ - \int_{a(Q^2)}^{a(Q_0^2)} \frac{\gamma(x) + \frac{d}{2}}{\beta(x)} dx \right], \quad (3.90)$$

where  $C(a(Q^2))$  is a function of the running coupling alone, and we have the renormalisation group functions

$$\beta(a) = -ba^2(1+ca+\dots), \quad (3.91)$$

$$\gamma(a) = -\gamma_0 a + \dots, \quad (3.92)$$

(see sub-section 1.3.2) for the condensate, where  $\gamma_0$  is its anomalous dimension. If one restricts oneself to the terms explicitly indicated in (3.91-3.92), the integral is straightforward, and as a result one obtains

$$G_0(a) = C\left(\frac{\mu^2}{Q^2}\right)^{d/2} \exp\left(\frac{d}{2ba(Q^2)}\right) a^\delta(Q^2) \left[ 1 + \mathcal{O}\left(\frac{1}{a(Q^2)}\right) \right], \quad (3.93)$$

where  $\delta = \frac{d}{2} \frac{c}{b} - \frac{\gamma_0}{b}$ , and  $C$  is a scale-independent constant which contains the truly nonperturbative information. If we admit that this constant is complex, say  $C = C_R + i\pi C_I$ , a general ambiguity of the form (2.43), originating from  $R^{PT}(Q^2)$ , may be cancelled by imposing the following identifications (we assume (2.20) here):

$$\gamma = -\delta, \quad (3.94)$$

$$z_i = -\frac{2b}{d}, \quad (3.95)$$

$$A = C_I \left(\frac{4}{b}\right)^\delta \Gamma(1-\delta) \frac{Q^8}{\mu^8}, \quad (3.96)$$

an idea first advanced in [56]. These relations fix the power and the position of the singularity, and the normalisation of the large order coefficients, respectively. Thus, a given *IR* renormalon can be cancelled by a condensate suggested from OPE principles. This illustrates the connection between perturbative and nonperturbative QCD that *IR* renormalons establish. The relation (3.95) can also be considered a further argument for the leading-*b* approximation, based in the RG as the arguments of subsections 3.2.1 and 3.2.3. In fact, as it has been extensively illustrated in the present Chapter, the fact that a given observable has singularities in the Borel transform at  $z \sim 1/b$  necessarily leads to its large-orders coefficients  $d_k$  containing a  $b^k$  term. As we have seen, this is a consequence of retaining only the leading term in the RG  $\beta$ -function.

### 3.3.2 Perturbative series and phenomenology

The more down-to-earth mind may ask at this point what exactly will be the visible effects of large-order behaviour and renormalons on the phenomenology. Obviously, in general only the first two or three coefficients in perturbative QED and QCD are known, and low order coefficients have no effect in determining the large-order behaviour (*e.g.*, the position of the singularities in the Borel plane). But, on the other hand, one may wonder if the lowest order exact coefficients are already consistent with the expectations for large orders. As a matter of fact, despite the arguments put forward on section 2.1 towards proving that QED perturbation series are hopelessly divergent, the numerical agreement between experiment and QED theory is rather good. For instance, the anomalous magnetic moment for the electron ( $g-2$ ) is probably the most well-measured number in the history of Physics. It has been calculated [81] to  $\mathcal{O}(a^4)$ :

Observable	$R_\tau$	GLS	R-ratio	$(g-2)_{th.}$
$d_1$	5.2	3.58	1.411	-0.33
$d_2$	26.4	19.0	-12.8	1.18
$d_3$	-	-	-	-1.41
$a$	0.112	0.089	0.038	$7.3 \times 10^{-3}$
$d_1 a^2$	0.064	0.028	$2 \times 10^{-3}$	$-1.75 \times 10^{-5}$
$d_2 a^3$	0.036	0.013	$7 \times 10^{-4}$	$4.59 \times 10^{-7}$
$d_3 a^4$	-	-	-	$-4 \times 10^{-9}$

Table 3.1: State-of-the-art QCD.

$$(g-2)_{theoretical} = a - 0.328478965a^2 + 1.181241456a^3 - 1.4092(384)a^4 + 4.396(42) \times 10^{-12}, \quad (3.97)$$

where  $a = \frac{e^2}{\hbar c}$  is the expansion parameter. Now, the fine structure constant  $\alpha = 2\pi a$  has been experimentally measured. Via the quantum Hall effect, its value was measured to be  $1/137.0360037(27)$ , thus leading to a theoretical prediction

$$(g-2)_{qH} = 1.1596521564(229) \times 10^{-12}, \quad (3.98)$$

and the experimental measurement of  $(g-2)$  for the electron [82] gives

$$(g-2)_{experimental} = 1.1596521884(43) \times 10^{-12}. \quad (3.99)$$

Thus we have a remarkable agreement of eight digits.

In QCD, there is no physical quantity candidate to such a good agreement as it was found for the QED anomalous magnetic momentum. Table 3.1 compares the first few exact perturbative calculations for some QCD observables [83] with the QED result just mentioned. The QCD observables are the  $R_\tau$  ( $\tau$  decay to hadrons), the GLS sum rules at  $\sqrt{s} = \sqrt{3}$  GeV and the R-ratio ( $e^+e^-$  to hadrons) at  $m_{Z^0} = 91$  GeV.

The growth in the size of the perturbative coefficients is seen to be clearly more dramatic in QCD than in QED. Moreover, the size of the terms in the perturbative



series decreases comparatively slower in QCD than in QED, even at the energy of the  $m_Z$ . A word of caution must anyhow be said: all of the QCD results are dependent on the renormalisation scheme, which can totally change not only the quantitative but also the qualitative aspects of Table 3.1. However, with the values given for the perturbative coefficients, one obtains theoretical predictions fairly close to the experimental observations.



# Chapter 4

## A reformulation of QCD perturbation theory

We now have a clear picture of the problems facing large-orders QCD perturbation theory. As was seen in the previous Chapter, the (renormalised) QCD perturbation theory coefficients have a large-orders growth which will be, in general, a linear combination of fixed-sign and alternating-sign factorials. This implies that any naive sum of perturbation theory terms will diverge, and therefore underlines the need for a resummation method. Several methods of resummation have already been discussed in Chapter 2. Of these, the Padé approximants resummations will predictably suffer from having singularities in a region where the coupling takes physical values, since all QCD observables have *IR* renormalons and this leads to poles on the positive real semi-axis. On the other hand, the Borel resummation procedure, besides being ambiguously defined whenever *IR* renormalons are present, seems *ad hoc* and it is not clear how the exact coefficients can be included in a natural way (the Padé approximants do not have this caveat), unless one includes them in the renormalised coupling to all-orders [84], a procedure which can be computationally cumbersome. So, we would like to have a resummation procedure based on RS-invariants, manageable on a order by order basis, and not suffering from the unavoidable instabilities that Padé approximants will display in the presence of *IR* renormalons.

The reformulation of QCD perturbation theory hereby described was suggested by C.J. Maxwell in [1], which the first section of this Chapter will follow closely.

## 4.1 Derivation of the truncated continued function method

We shall start by considering the perturbation theory expansion of a generic and dimensionless QCD observable:

$$R(Q) = a + d_1 a^2 + d_2 a^3 + \dots d_k a^{k+1} + \dots \quad (4.1)$$

Such observables are known as *effective charges*. These satisfy several important properties. The first one is Asymptotic Freedom (AF; see Chapter 1), which can be stated simply as

$$\lim_{Q \rightarrow +\infty} R(Q) = 0. \quad (4.2)$$

To see how the second property comes about, one starts by going back to the effective charge  $\beta$ -function equation,

$$\frac{dR}{d(b \ln Q)} = -\rho(R) \equiv -R^2(1 + cR + \rho_2 R^2 + \dots \rho_k R^k + \dots). \quad (4.3)$$

On integrating this equation, and imposing AF as a boundary condition, one has

$$b \ln \frac{Q}{\Lambda_R} = - \int_0^{R(Q)} \frac{1}{\rho(x)} dx + (infinity), \quad (4.4)$$

where  $\Lambda_R$  is a finite constant dependent on the way the infinity is defined. The later may be chosen to be

$$(infinity) = \int_0^{+\infty} \frac{1}{\eta(x)} dx, \quad (4.5)$$

where  $\eta(x)$  must have the same behaviour as  $\rho(x)$  when  $x \rightarrow 0$ . We choose  $\eta(x) = x^2(1 + cx)$  (as suggested in [14]), and then one can rearrange (4.4) as

$$b \ln \frac{Q}{\Lambda_R} = \int_{R(Q)}^{+\infty} \frac{1}{x^2(1 + cx)} dx + \int_0^{R(Q)} \left[ \frac{1}{x^2(1 + cx)} - \frac{1}{\rho(x)} \right] dx. \quad (4.6)$$

The first integral on the right-hand side gives just the customary  $F(R(Q))$  ( $F(x)$  having already been defined in (1.38)), whereas the second is defined to be  $\Delta\rho_0(Q)$ ,

$$\Delta\rho_0(Q)\equiv\int_0^{R(Q)}\left[\frac{1}{x^2(1+cx)}-\frac{1}{\rho(x)}\right]dx, \quad (4.7)$$

such that one can write (see (1.53))

$$F(R(Q))=\rho_0(Q)-\Delta\rho_0(Q). \quad (4.8)$$

It must be noted that AF implies  $\Delta\rho_0(Q)\rightarrow 0$  when  $Q\rightarrow +\infty$ . Therefore, one has

$$\lim_{Q\rightarrow +\infty} Q\mathcal{F}(R(Q))=\Lambda_R, \quad (4.9)$$

where

$$\mathcal{F}(x)\equiv e^{-\frac{1}{bx}}\left(1+\frac{1}{cx}\right)^{c/b}, \quad (4.10)$$

and  $-b\ln(\mathcal{F}(x))=F(x)$ . The property (4.9) is termed *Asymptotic Scaling* (AS)<sup>1</sup>. It could be used to test QCD at large values of  $Q$ , but since  $\Lambda_R$  is observable-dependent, it is not useful at fixed values of  $Q$ . Fortunately, we can transform it into a more useful form provided the NLO coefficient ( $d_1\equiv d_1^{\overline{MS}}(\mu=Q)$ ) has been calculated. Then, one can convert  $\Lambda_R$  into a RS dependent (but observable-independent) scaling constant via the *exact* Celmaster-Gonsalves relation (1.54) [13]. Thus, we can rewrite (4.9) as *Universal Asymptotic Scaling* (UAS):

$$\lim_{Q\rightarrow +\infty} Q\mathcal{F}(R(Q))e^{-d_1^{\overline{MS}}/b}=\tilde{\Lambda}_{\overline{MS}}. \quad (4.11)$$

It must be noted that  $d_1^{RS}$  is independent of  $Q$ , and that  $\Lambda_{RS}$  is only required to be consistent with the renormalisation scheme used to calculate  $d_1$ . The data of various observables can now be used to test QCD at fixed energy by looking at the scatter of the function on the left-hand side of (4.11) [14, 15]. Of course, at finite  $Q$  UAS is violated, and therefore one will have

$$Q\mathcal{F}(R(Q))e^{-d_1^{\overline{MS}}/b}=\tilde{\Lambda}_{\overline{MS}}e^{\frac{\Delta\rho_0(Q)}{b}}\simeq\tilde{\Lambda}_{\overline{MS}}\left(1+\frac{\Delta\rho_0(Q)}{b}\right). \quad (4.12)$$

---

<sup>1</sup>The terminology and the property itself are used in Lattice Gauge Theory to assess how close the lattice coupling is to its continuum behaviour at a given value of lattice spacing. The scaling results as the inverse lattice spacing goes to infinity, which corresponds to take the ultraviolet limit.

Since it will regulate the violations of UAS, it is therefore of importance to know the behaviour of  $\Delta\rho_0(Q)$ . With this goal in mind, one expands the integrand in (4.7) and integrates term by term, obtaining:

$$\Delta\rho_0(Q) = \rho_2 R + (\rho_3 - 2c\rho_2) \frac{R^2}{2} + \dots \quad (4.13)$$

The important fact to retain here is that the leading term in the sub-asymptotic scaling violation effects  $\Delta\rho_0(Q)$  will be proportional to  $\rho_2$ . Furthermore, it is important to recognise that one can define a new effective charge such that  $\Delta\rho_0(Q) \equiv \rho_2 R^{(1)}(Q)$ , parameterising the leading behaviour of the sub-asymptotic scaling violation effects. The formal expansion of this new effective charge  $R^{(1)}(Q)$  can be obtained replacing (4.1) in (4.13). The result is

$$R^{(1)}(Q) = a + (d_1 + \frac{\rho_3}{2\rho_2} - c)a^2 + \dots, \quad (4.14)$$

and from the expansion (4.14) one can extract the new  $d_1^{(1)}, d_2^{(1)}, \dots$ . One now notes that  $R^{(1)}(Q)$ , being an effective charge, will also obey the UAS property, and its sub-asymptotic violating effects will be proportional to yet another effective charge  $R^{(2)}(Q)$ , and so on *ad infinitum*, generating a self-similar infinite construction. The only changes at each level are in the coefficients of the formal expansions of the successive effective charges, and consequently in the corresponding RS-invariants. For instance,

$$\rho_0^{(1)} = b \ln \frac{Q}{\Lambda_{\overline{MS}}} - d_1^{(1)} = \rho_0 - \frac{\rho_3}{2\rho_2} + c, \quad (4.15)$$

$$\rho_2^{(1)} = \rho_2 + d_2^{(1)} - c d_1^{(1)} - d_1^{(1)2}, \quad (4.16)$$

$$\rho_3^{(1)} = \rho_3 + 2d_3^{(1)} - 2\rho_2 d_1^{(1)} + c d_1^{(1)2} + 4d_1^{(1)3} - 6d_1^{(1)} d_2^{(1)}, \quad (4.17)$$

$$\vdots \quad \vdots$$

where  $c_2^{(1)} = \rho_2^{(0)}, c_3^{(1)} = \rho_3^{(0)}, \dots$  because we have chosen to work in the EC scheme at every step.

We note in passing that the general formula

$$\rho_0^{(i+1)} = \rho_0 - \sum_{j=0}^i \frac{\rho_3^{(j)}}{2\rho_2^{(j)}} + (i+1)c \quad (4.18)$$

holds for every  $i \geq 0$ .

Every effective charge  $R^{(i)}$  will obey an equation of the form (4.8), which will depend on a new effective charge  $R^{(i+1)}$ . Explicitly,

$$F(R(Q)) = \rho_0(Q) - \rho_2 R^{(1)}(Q), \quad (4.19)$$

$$F(R^{(1)}(Q)) = \rho_0^{(1)}(Q) - \rho_2^{(1)} R^{(2)}(Q), \quad (4.20)$$

$$F(R^{(2)}(Q)) = \rho_0^{(2)}(Q) - \rho_2^{(2)} R^{(3)}(Q), \quad (4.21)$$

$$\vdots \quad \vdots$$

$$F(R^{(n)}(Q)) = \rho_0^{(n)}(Q) - \rho_2^{(n)} R^{(n+1)}(Q), \quad (4.22)$$

$$\vdots \quad \vdots$$

Using the inverse of  $F(x)$  defined in (1.42), we can solve this infinite chain of equations for the first effective charge:

$$R(Q) = G(\rho_0 - \rho_2 G(\rho_0^{(1)} - \rho_2^{(1)} G(\rho_0^{(2)} - \rho_2^{(2)} G(\dots))))). \quad (4.23)$$

This is the *continued function* representation of our observable. In realistic problems, one has to truncate it at some finite order, but, if the successive effective charges become smaller and smaller, the construction obtained by truncating (4.23) at successively higher orders will be convergent. It must be noted that a given  $\rho_0^{(n)}$  involves all the  $\rho_k$  for  $k \leq 2n+1$ , thus requiring a  $N^{2n+1}LO$  perturbative calculation, and  $\rho_2^{(n)}$  involves all the  $\rho_k$  for  $k \leq 2n+2$ , thus requiring a  $N^{2n+2}LO$  perturbative calculation for  $R(Q)$ . In other words, the successive truncations of (4.23),

$$G(\rho_0), \quad (4.24)$$

$$G(\rho_0 - \rho_2 G(\rho_0^{(1)})), \quad (4.25)$$

$$G(\rho_0 - \rho_2 G(\rho_0^{(1)} - \rho_2^{(1)} G(\rho_0^{(2)}))), \quad (4.26)$$

$$\vdots$$

$$G(\rho_0 - \rho_2 G(\dots - \rho_2^{(n-1)} G(\rho_0^{(n)}))), \quad (4.27)$$

$$\vdots$$

only require knowledge of the  $\mathcal{O}(a), \mathcal{O}(a^3), \mathcal{O}(a^5), \dots \mathcal{O}(a^{2n+1}), \dots$  calculations, and therefore jump orders in perturbation theory. To be able to compare order by order with other methods, one creates the equivalent of the orders  $\mathcal{O}(a^2), \mathcal{O}(a^4), \dots \mathcal{O}(a^{2n+2}),$

$$G(\rho_0 - \rho_2 G(\rho_0)), \quad (4.28)$$

$$G(\rho_0 - \rho_2 G(\rho_0^{(1)} - \rho_2^{(1)} G(\rho_0^{(1)}))), \quad (4.29)$$

$\vdots$

$$G(\rho_0 - \rho_2 G(\dots - \rho_2^{(n)} G(\rho_0^{(n)}))), \quad (4.30)$$

$\vdots$

## 4.2 CORGI in the leading- $b$ approximation

Our explorations of large-orders QCD, as can be seen from the previous Chapter, rely heavily on the leading- $b$  approximation, where one considers  $c \approx 0$ . As a consequence, one will not need the full perturbative series in (1.81) or (1.87). In fact, we shall use the CORGI series in the leading- $b$  approximation. With  $c=0$ , we note that the sum (1.90) becomes a geometric series

$$a_0 \equiv a + d_1 a^2 + d_1^2 a^3 + d_1^3 a^4 + d_1^4 a^5 + \dots \left( = \frac{a}{1 - d_1 a} \right). \quad (4.31)$$

The neat thing about this form of  $a_0$  is that its inverse  $a = \frac{a_0}{1 + d_1 a_0}$  can be easily expanded. Thus, by replacing its expansion in the original power series (1.45), one obtains

$$R(a_0) = a_0 + (d_2 - d_1^2) a_0^3 + (d_3 - 3d_1 d_2 + 2d_1^3) a_0^4 + (d_4 - 4d_1 d_3 + 6d_1^2 d_2 - 3d_1^4) a_0^5 + \dots \quad (4.32)$$

The coefficients in this series are actually the RS-invariants of sub-sections 1.4.8 and 1.4.9, with the slight difference that here  $X_k = X_k(c=0, c_2=0, c_3=0, \dots c_k=0, \dots)$ . Thus, it becomes computationally very easy to generate the RS-invariants for the CORGI scheme series in the leading- $b$  approximation.

The advantages of basing perturbation theory in the CORGI RS-invariants  $X_k$  are: first, that the first coefficient  $d_1$  (known for most cases of interest) can be absorbed

into the two-loop coupling; secondly, that the known NNLO, N<sup>3</sup>LO... calculations (at the time of writing, this only applies to  $X_2$ ) can be included exactly; thirdly, that any high-order coefficients, usually available in the leading- $b$  approximation, can be dealt with in the form of renormalisation scheme invariants. One further important advantage of using the  $X_k$ 's in the leading- $b$  approximation is that the specific construction (4.32) can be easily generated from a geometrical series expansion, whereas the use of the  $c$ -dependent  $X_k$ 's or  $\rho_k$ 's becomes computationally cumbersome for  $k > 20$ . Thus, the formulation described in this sub-section is specially suited for the purpose of studying the large-order behaviour of QCD in the leading- $b$  approximation.

As it was advanced in sub-section 1.4.9, part of the motivation of the CORGI method is that it allows us to get rid of the unphysical dependence on the scale  $\mu$ , and obtain the correct physical dependence on the energy of the observable. Indeed, if one uses for the time being the one loop coupling (1.35) as  $a(\mu)$ , one notices that, since by virtue of (1.53) one has

$$d_1^{RS} = \left( b \ln \frac{\mu}{\Lambda_{RS}} - b \ln \frac{Q}{\Lambda_R} \right), \quad (4.33)$$

consequently the sum (4.31) can be written as

$$\frac{a(\mu)}{1 - \left( b \ln \frac{\mu}{\Lambda_{RS}} - b \ln \frac{Q}{\Lambda_R} \right) a(\mu)} = \frac{1}{b \ln \frac{Q}{\Lambda_R}}. \quad (4.34)$$

Thus the unphysical logarithms  $\ln \frac{\mu}{\Lambda_{RS}}$  cancel with  $a(\mu)$ , building the correct  $\ln \frac{Q}{\Lambda_R}$  dependence. This coupling  $a_0$  is renormalisation scale independent and satisfies the property of Asymptotic Freedom independently of the value of the constant  $\Lambda_R$ , whereas in the traditional approach one would have the truncated expansion

$$a(\mu) + d_1(\mu)a^2(\mu) \longrightarrow a(xQ) + d_1(xQ)a^2(xQ), \quad (4.35)$$

which is obviously dependent on  $x$ . This underlines the advantage of expressing the coupling in terms of the physically meaningful ratio  $\frac{Q}{\Lambda_R}$ .

### 4.3 Study of one example on the quest for a proof of convergence

It was claimed in [1] that the truncated continued function construction would converge to the Borel sum if only  $UV$  renormalons were present. This claim was substantiated by noting that the one loop version of  $G(x)$ , which is  $G'(x) = \frac{1}{x}$ , gives rise to a *continued fraction* instead of a continued function such as (4.23),

$$R(a) = \frac{1}{\rho_0 - \frac{\rho_2}{\rho_0^{(1)} - \frac{\rho_2^{(1)}}{\rho_0^{(2)} - \dots \frac{\rho_2^{(n-1)}}{\rho_0^{(n)}}}}}. \quad (4.36)$$

The successive truncations of this continued fraction give us the correspondent of the “natural” sub-sequence. That (4.36) is exactly the same as the associated form (2.55) of the continued fractions representation of the series  $R(a)$ , that is, the diagonal sequence of Padé approximants, was the original motivation for the truncated continued function method. Note that, for instance, for the simple alternating-factorial series  $d_k = (-1)^k k!$ , one has  $\rho_0^{(n)} = \frac{1}{a} - d_1 + 2n$  and  $\rho_2^{(n)} = (n+1)^2$  ( $d_k = k!$  would differ by having  $\rho_0^{(n)} = \frac{1}{a} - d_1 - 2n$ )<sup>2</sup>. The very same coefficients can be obtained for the associated continued fraction with the  $K_j$ ’s in the first column of Table 2.3. From the arguments of Chapter 2, one can expect convergence for the one loop  $G'(x)$  in this simple case. However, it goes without saying that the limit of this continued fraction may not be the same as the limit of the two-loop  $G(x)$ . Also, the convergence of these continued fractions proves nothing where the convergence of the full, two-loop  $G(x)$  is concerned. Indeed, since  $\lim_{c \rightarrow 0} \text{cln} \left| \frac{cx}{1+cx} \right| = 0$ , one could naively expect to motivate this continued fractions approach by considering the power series expansion of  $G(x)$  in powers of  $c$ , which should converge to  $G'(x)$  on the limit  $c \rightarrow 0$ . However, it is not possible to study this limit because  $\lim_{c \rightarrow 0^+} G^+(x, c) = +\infty$ , and

---

<sup>2</sup>This is just a simple numerical example. One is not considering a CORGI resummed series, nor is one imposing the leading- $b$  approximation.



we would therefore be expanding around a special point.

To address the convergence of the continued function approach, we shall consider one more time the Stieltjes series

$$R(a) = a - a^2 + 2!a^3 - 3!a^4 + \dots (-1)^k k! a^{k+1} + \dots \quad (4.37)$$

The CORGI resummed version of this series is

$$R(a_0) = a_0 + X_2 a_0^3 + X_3 a_0^4 + \dots X_k a_0^{k+1} + \dots, \quad (4.38)$$

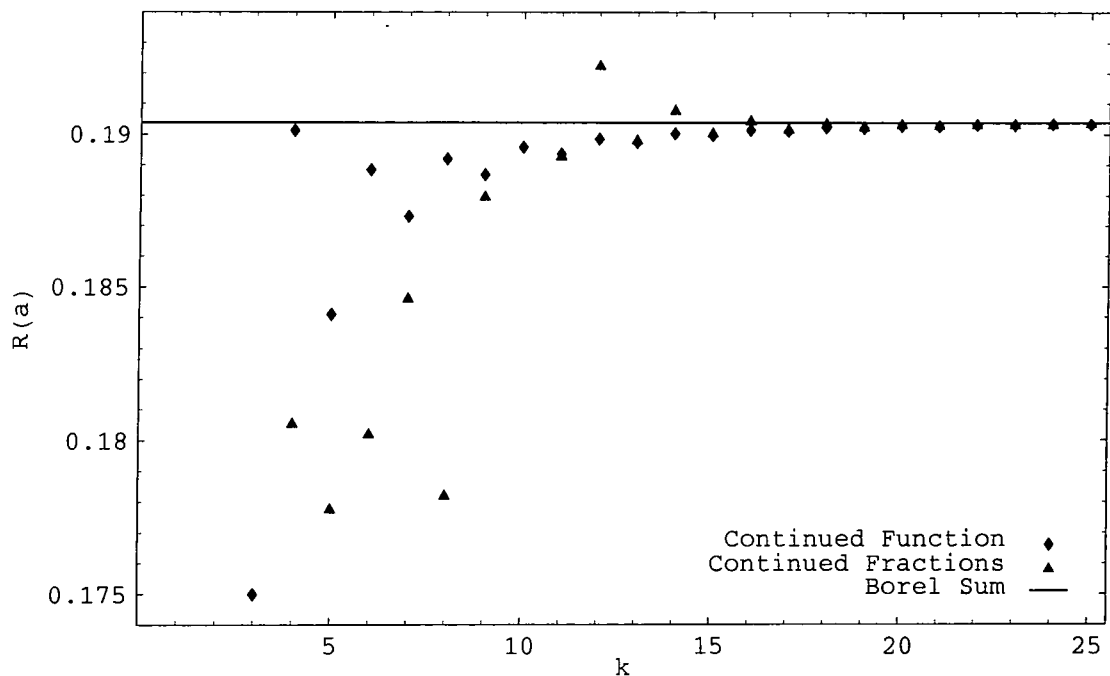


Figure 4.1: Resummation of a single  $UV_1$  renormalon CORGI series using the truncated continued function method and Padé approximants.

where  $a_0$  is given by (1.91) with  $Q = 1.5$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 320$  MeV,  $N_f = 3$  flavours, the  $X_k$ 's are better calculated from (4.37) with the algorithm discussed in Appendix E, and the leading- $b$  approximation is used ( $X_k = X_k(d_1^{(1)}, \dots, d_k^{(k)})b^k$ ). The resummation of this series with both the truncated continued function method and the Padé approximants normal sequence is shown in Figure 4.1. Both methods seem to

converge to the Borel sum, even if the agreement is only of three digits for  $\mathcal{O}(a^{26})$ , for both methods, at the low energy chosen. Thus, the limit of the two methods seems to be virtually indistinguishable in the case when  $R(a)$  is a simple series. If one looks at the odd orders of the continued function truncations in Figure 4.1 (which reproduce the “natural” sub-sequence (4.24-4.27)), it is clear that they are converging to the Borel sum like a saturating exponential. The same can be said of the diagonal sequence of Padé approximants. (The “interpolation” sub-sequence provides, at each order, a positive “fluctuation” of variable size on the previous value of the “natural” sub-sequence, therefore not improving at all the convergence.) This is a remarkable feature, and it reassures us about the convergence of at least one sub-sequence of the truncated continued functions. However, it has to be said that the CORGI series  $R(a_0)$  is *not* a *Stieltjes series*, even if the Padé approximants appear to converge to the Borel sum (which is a Stieltjes function), and effectively  $X_k \sim (-1)^k k!$ . In fact, the continued fraction coefficients  $K_j$  for series (4.38) take negative values for some values of  $j$  ( $K_6, K_7; K_{22}, K_{23}; K_{50}, K_{51}; \dots$ ).

Nevertheless, that the “natural” sub-sequence of truncated continued functions converges if the series is a CORGI resummed series of a single  $UV_1$  “renormalon” can be understood as follows.

Given the fact that numerical evidence shows that  $\rho_0^{(n)} \sim n$  and  $\rho_2^{(n)} \sim n^2$  in this case, it follows first that  $G(\rho_0^{(n)}) > 0, \forall_n$ , and secondly that  $\rho_2^{(n)} G(\rho_0^{(n+1)}) > 0, \forall_n$ . Therefore, one has trivially

$$G(\rho_0^{(n)}) < G(\rho_0^{(n)} - \rho_2^{(n)} G(\rho_0^{(n+1)})), \forall_n. \quad (4.39)$$

Using the same reasoning at every order down to  $\rho_0^{(0)}$ , one can show that the successive truncations of the “natural” sub-sequence of continued functions are an *increasing sequence*:

$$G(\rho_0^{(0)} - \dots G(\rho_0^{(n)})) < G(\rho_0^{(0)} - \dots G(\rho_0^{(n)} - \rho_2^{(n)} G(\rho_0^{(n+1)}))), \forall_n. \quad (4.40)$$

Of course, in this step one has assumed that  $G^+(z)$  will always be the chosen branch of the function  $G(z)$  at every level. That it is so for the last  $G$ , does not need any proof. For the penultimate level in a given truncation, it is also easy to see that  $\rho_0^{(n)} - \rho_2^{(n)} G(\rho_0^{(n+1)}) > 0, \forall_n$ . (This only requires checking numerically that  $\mathcal{O}(\frac{1}{n}) >$

$G(\mathcal{O}(n))$ .) However, it remains an open problem how to prove that  $G^+(z)$  is chosen at all levels in a given truncation. Anyway, this assumption is plausible if one notices that

$$\mathcal{O}\left(\frac{1}{n}\right) > G^+(\mathcal{O}(n) - \mathcal{O}(n^2)G^+(\mathcal{O}(n) \dots)), \quad (4.41)$$

and thus

$$\rho_0^{(n)} > \rho_2^{(n)} G(\rho_0^{(n+1)} - \rho_2^{(n+1)} G(\rho_0^{(n+2)} \dots)), \quad (4.42)$$

rendering the argument of  $G$  at any given level presumably positive.

As it happens that  $G(\rho_0^{(n)}) \rightarrow 0$ , we also know from the properties of  $G$  that  $\rho_2^{(n)} G(\rho_0^{(n+1)}) \rightarrow 0$ . It is then intuitive that the contributions of higher and higher orders shall be smaller and smaller. This can also be understood by defining

$$\mathcal{G}_n(x) \equiv G(\rho_0^{(0)} \dots \rho_2^{(n-1)} G(x)), \quad (4.43)$$

such that we can look at the difference

$$G(\dots G(\rho_0^{(n+1)})) - G(\dots G(\rho_0^{(n)})) \left( = \mathcal{G}_{n+1}(\rho_0^{(n+1)}) - \mathcal{G}_n(\rho_0^{(n)}) \right), \quad (4.44)$$

by assuming that  $\mathcal{G}_n(x)$  has a regular point at  $x = \rho_0^{(n)}$ , so that we can expand  $\mathcal{G}_n(x)$  in a series around this point, and thus rewrite (4.44) as

$$\mathcal{G}_n(\rho_0^{(n)} - \rho_2^{(n)} G(\rho_0^{(n+1)})) - \mathcal{G}_n(\rho_0^{(n)}) = \frac{\partial \mathcal{G}_n(x)}{\partial x} \Big|_{x=\rho_0^{(n)}} \left( -\rho_2^{(n)} G(\rho_0^{(n+1)}) \right) + \mathcal{O}\left( \rho_2^{(n)} G(\rho_0^{(n+1)}) \right)^2. \quad (4.45)$$

Therefore, it is plausible that the difference between successive truncations will vanish on the limit  $n \rightarrow \infty$ . Thus we have shown with rather weak assumptions that in the case at hand: a) the “natural” sub-sequence is increasing, and b) each new order brings increasingly smaller contributions to the final result.

It must be noted that these considerations do not give any guarantee that the limit of convergence (if there is one) will be the Borel sum. Also, in realistic problems, the large-order growth always contains both *UV* and *IR* renormalons, and, as a consequence, it cannot be guaranteed that the higher-order  $\rho_0^{(n)}$ 's will remain positive. Therefore,  $G^-(x)$  intervenes in realistic problems.

In general cases, the fundamental criterion to assess the convergence of the method will still be the behaviour of  $\rho_0^{(n)}$  and, secondarily,  $\rho_2^{(n)}$ . As it was mentioned, for a series with a single fixed-sign factorial, one has  $\rho_0^{(n)} \sim -n$ , whereas  $\rho_2^{(n)} \sim n^2$ . More generally, whenever the predominant renormalon in the observable is  $IR$  (e.g.,  $IR_1+IR_2+UV_2$ ), one expects a linearly decreasing sequence for  $\rho_0^{(n)}$ . If the predominant renormalon is  $UV$  (e.g.,  $UV_1+UV_2+IR_2$ ), one expects a linearly increasing sequence for  $\rho_0^{(n)}$ . The  $\rho_2^{(n)}$ 's are less important for the convergence properties, as  $\rho_2^{(n)} \sim n^a$  ( $a \in [2, 3]$ ) in general, and this is tamed by  $G(\rho_0^{(n+1)})$ . One may ask what happens when we have a “balanced” combination of renormalons (say,  $UV_1+UV_2+IR_1+IR_2$ ). Well, it turns out that  $\rho_0^{(n)} \sim \text{constant}$  in those cases. How an almost constant sequence of  $\rho_0^{(n)}$ 's or a sequence taking negative values will affect the asymptotic result is not very clear at this point.

## 4.4 Summary

A new method to resum QCD perturbation theory was described in this Chapter. It is based on RS-invariants, it allows us to include exact calculations order by order, and it does not seem *a priori* to be irremediably flawed when  $IR$  renormalons are present. Also, being based on the physical property of Universal Asymptotic Freedom, it is not an *ad hoc* procedure as the Borel resummations. Furthermore, it reformulates QCD perturbation theory avoiding the unphysical concept of coupling constant, which is traded as a fundamental parameter for the first RS-invariant, which has the correct physical dependence on  $\frac{Q}{\Lambda_R}$ . The concept of perturbation theory expansion in powers of the coupling is superseded by a construction originating from the renormalisation group, and thus theoretically sound.

We shall now turn to the exploration of this method for some observables of interest.

## Chapter 5

# Reformulated perturbative QCD for some euclidean observables

The goal of this Chapter will be to apply the reformulation of perturbative QCD described in the previous Chapter to the study of some euclidean physical observables. The available exact results will be used where possible, but we shall be mainly interested in how this novel method copes with the (divergent) large-orders behaviour of perturbative QCD. Comparisons will be made with other resummation methods. These will be what we shall call “naive perturbation theory”, Padé approximants, and the Borel sum. “Naive perturbation theory” consists simply of adding up the perturbation series, the series to be summed being in our case a CORGI scheme resummed series. Padé approximants, which are at present a subject of much interest in the literature, are an order-by-order method of resumming power series, and consist of recasting a power series as a quotient of polynomials. An important issue concerning Padé approximants is the fact that they reduce the renormalisation scheme dependence of the QCD calculations [36, 37, 38]. The Borel sum is an all-orders method which, as we have seen in Chapter 3, can be very useful to study the structure of the large-orders coefficients of perturbation theory, by classifying its Borel transform singularities. Borel summation methods have been extensively pursued in the literature [50, 60, 75, 84]. Padé approximants have connections with both Borel summations and the truncated continued functions method. As we have seen in Chapter 2, the Padé approximants of Stieltjes series converge to the Borel sum. However,  $IR$  renormalons are present in real-world

examples, and it remains an open question how they will affect the convergence of Padé approximants. The truncated continued function method was shown to be, at least formally, close to Padé approximants, since for  $c = 0$  the continued function construction becomes a continued fraction, and thus a Padé approximant. An interesting question is therefore if the numerical results of the truncated continued function method are in any way qualitatively similar to the ones obtained with Padé approximants. However, as stated before, there is no *a priori* reason for the limit chosen by the truncated continued function method to be the Borel sum, even when only Stieltjes series are considered.

## 5.1 The Adler function revisited

The Adler function perturbation theory expansion has been defined in equation (3.4), and the series that we will be dealing with will be the one in (3.5) for  $D(a)$ . The conventional approach to this series is simply to add it up term-by-term. This is what is normally done in the literature when only the two or three exactly known coefficients are considered. We shall consider a CORGI resummed version of series (3.5), and that is what we shall call “naive perturbation theory”. However, we have seen in Chapter 2 and Chapter 3 that a QCD perturbative series is bound to diverge. Moreover, we will be considering the renormalons contribution to the large-orders coefficients, and these are enough to lead to what is, at best, an asymptotically convergent series, since the coefficients will include alternating-sign factorials and fixed-sign factorials. Thus, consideration of the “naive perturbation theory” series will only show us where and how the ordinary perturbation theory will diverge, underlining the need for a resummation procedure.

In this section, we shall apply several resummation methods to the study of the Adler function  $D(a)$ . This will provide a first taste of the concrete problems that arise in large-orders QCD, and how techniques that will later be used for other observables will be applied.

### 5.1.1 The CORGI series for the Adler function $D(a)$

The first two exact coefficients of the Adler function  $D(a)$  have been calculated [73, 74] in the  $\overline{MS}$  scheme with  $\mu=Q$ :

$$d_1^{\overline{MS}} = \left(-\frac{11}{12} + \frac{2}{3}\zeta_3\right)N_f + N\left(\frac{41}{8} - \frac{11}{3}\zeta_3\right) - \frac{1}{8}C_F, \quad (5.1)$$

$$\begin{aligned} d_2^{\overline{MS}} = & \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right)N_f^2 + N\left(-\frac{970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5\right)N_f + C_F\left(-\frac{29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5\right)N_f \\ & + N^2\left(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5\right) + N\left(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5\right)C_F - \frac{23}{32}C_F^2. \end{aligned} \quad (5.2)$$

At the time of writing, these are all the exact coefficients known.

Since we want to resum as much information as possible, we shall implement Complete Renormalisation Group Improvement (CORGI) at every order. As a consequence, one will have a series with RS-invariants  $X_k$  as its coefficients (see sub-section 1.4.9). The  $X_2$  in the said series will be exact,

$$X_2 = d_2^{\overline{MS}} - c d_1^{\overline{MS}} - d_1^{\overline{MS}^2} + c_2^{\overline{MS}}, \quad (5.3)$$

since we have all the information needed to calculate it ( $d_1^{\overline{MS}}$ ,  $d_2^{\overline{MS}}$ ,  $c_2^{\overline{MS}}$ ). (The renormalisation scheme invariant  $X_2$  will also be known exactly for the other observables of interest.) The higher orders CORGI RS-invariants  $X_k$  ( $k \geq 3$ ) cannot be calculated exactly, since  $d_3^{RS}$  is already not known. We shall use the leading- $b$  renormalon coefficients  $d_k^{(k)}$  ( $k \geq 1$ ) in the V-scheme (see (3.29)) for these. As a consequence, it is enough to use the leading- $b$  RS-invariants of sub-section 4.2 that can be easily computed up to very high orders:

$$X_3^{(3)} = d_3^{(3)} - 3d_1^{(1)}d_2^{(2)} + 2d_1^{(1)^3}, \quad (5.4)$$

$$X_4^{(4)} = d_4^{(4)} - 4d_1^{(1)}d_3^{(3)} + 6d_1^{(1)^2}d_2^{(2)} - 3d_1^{(1)^4}, \quad (5.5)$$

$$\vdots \quad \quad \quad \vdots$$

Thus, the series to be summed is

$$D(a_0) = a_0 + X_2 a_0^3 + X_3^{(3)} b^3 a_0^4 + X_4^{(4)} b^4 a_0^5 + \dots X_k^{(k)} b^k a_0^{k+1} + \dots, \quad (5.6)$$

and the leading- $b$  approximation is made on the RS-invariants  $X_k^{(k)}$  rather than on the coefficients  $d_k^{(k)}$  (we define  $X_k^{(L)} \equiv X_k^{(k)} b^k$  for future reference).  $a_0(Q)$  is the two-loop coupling (1.91) in the  $\overline{MS}$  scheme at the EC scale  $\mu = \exp(-d_1^{\overline{MS}}/b)Q$ , and this concludes the specification of the CORGI scheme.

Since the Adler function  $D(a)$  large-orders coefficients in the V-scheme are numerically dominated by the leading renormalon  $UV_1$  [60], we expect the series (5.6) to be also, basically, an alternating-sign factorial. That this is a truthful conclusion can be checked by way of numerical experiments. However, one knows that (5.6) is definitely not a Stieltjes series, because  $IR$  renormalons are present. One could ask, nevertheless, if its  $UV$  renormalons (which one knows to be dominant in the V-scheme), considered separately, constitute a Stieltjes series. As we have shown in sub-section 3.1.4, it is not so.

The series (5.6) shall be the one summed up in the “naive perturbation theory” approach, resummed in the Padé approximants method, and used in the calculation of the continued function method RS invariants  $\rho_0^{(k)}$  and  $\rho_2^{(k)}$ , these later quantities being calculated conveniently using the algorithm of Appendix E.

### 5.1.2 The Borel sum of the Adler function $D(a)$

The integrals in equation (3.39) can be evaluated in terms of the exponential integral function  $Ei(x)$  defined in Appendix A. The integrals which correspond to the  $IR$  renormalons have poles along the range of integration, and therefore require the specification of a prescription to go around these poles. This prescription will be the Cauchy Principal Value. Since we only need to consider a finite number of renormalons for any practical purpose, the exchange of the order of summation and integration is justified, and as a consequence we obtain the Borel sum of the Adler function  $D(a)$  as a simple sum of exponential integral functions (here shown in the V-scheme):

$$\begin{aligned}
D^{UV}(x) &= \sum_{\ell=1}^{+\infty} z_{\ell} \left\{ e^{\frac{z_{\ell}}{x}} Ei\left(-\frac{z_{\ell}}{x}\right) \left[ \frac{z_{\ell}}{x} (A_0(\ell) - z_{\ell} A_1(\ell)) - z_{\ell} A_1(\ell) \right] \right. \\
&\quad \left. + A_0(\ell) - z_{\ell} A_1(\ell) \right\}, \tag{5.7}
\end{aligned}$$



$$\begin{aligned}
D^{IR}(x) &= e^{-\frac{z_2}{x}} z_2 B_0(2) Ei\left(\frac{z_2}{x}\right) + \sum_{\ell=3}^{+\infty} z_\ell \left\{ e^{-\frac{z_\ell}{x}} Ei\left(\frac{z_\ell}{x}\right) \left[ \frac{z_\ell}{x} (B_0(\ell) + z_\ell B_1(\ell)) - z_\ell B_1(\ell) \right] \right. \\
&\quad \left. - B_0(\ell) - z_\ell B_1(\ell) \right\}.
\end{aligned} \tag{5.8}$$

The expressions above only include the large-order behaviour of the Adler function  $D(a)$  in the leading- $b$  approximation, that is, the renormalonic content. If comparisons with order-by-order reformulations of QCD perturbation theory series CORGI resummed are to have any meaning, one should also include the known exact coefficients. This problem, although for the EC scheme Adler function  $D(a)$ , has been dealt with in the past [75, 84] by renormalising the coupling to as many orders as possible with the EC  $\beta$ -function. In that approach, the exact coefficients were included in the RS-invariant  $\rho_2$ , and the remaining  $\rho_k$ 's were calculated with the large-order leading- $b$  coefficients  $d_k^{(k)}$ . The renormalisation of the coupling then amounts to solve numerically equation (1.30) for the coupling. This procedure is computationally cumbersome. Now, if we wanted to work in the V-scheme, we would have to evaluate  $D(a^V)$ . In our case, since we choose to work in the CORGI scheme, the coupling is the two-loop coupling  $a_0$  (1.91) at the EC scale, and, recalling that  $a^V = \frac{a_0}{1+d_1^V a_0}$  (see (4.31)), we can transform into the CORGI scheme via

$$\frac{1}{a^V} \rightarrow \frac{1}{a_0} + d_1^V, \tag{5.9}$$

where  $d_1^V \equiv d_1^{(1)} b$ . One final circumstance to consider is that the series which the CORGI scheme Borel sum is summing differs from (5.6) because it includes  $X_2^{(L)}$  (calculated using the leading- $b$  coefficients  $d_k^{(k)}$ ) rather than  $X_2$  (exact). So, the contribution from the exact coefficients at order  $a_0^3$  still has to be included. This can be done by adding a  $(X_2 - X_2^{(L)}) a_0^3$  correction. Thus, the Borel sum of the CORGI series is

$$D\left(\frac{a_0}{1+d_1^{(V)} a_0}\right) + (X_2 - X_2^{(L)}) a_0^3, \tag{5.10}$$

where  $D(x)$  is the sum of the two functions in (5.7-5.8).

### 5.1.3 Padé approximants and QCD

Padé approximants have been a subject of much interest in the recent literature concerning resummations in perturbative QCD. The initial works by M.A. Samuel and others concentrated on using Padé approximants to predict the next perturbative coefficient in a QCD (or QED) perturbative series (see [85] and references therein). This is done calculating a Padé approximant of suitable order with the first few known perturbative coefficients  $d_1, \dots, d_k$ , then expanding the Padé approximant as a Taylor series, and truncating one order above, thus obtaining an estimated  $d_{k+1}^{est.}(d_1, \dots, d_k)$ . This program was pursued for the anomalous magnetic moment of the  $\tau$  lepton, and for the QCD sum rules. The agreement with exactly known terms or other estimation methods was found to be remarkably good [85]. However, the most spectacular prediction to be found was for the four-loops coefficient of the  $\beta$ -function [86], which was predicted to be

$$\beta_3^{\overline{MS}} \simeq 23600(\pm 900) - 6400(\pm 200)N_f + 350(\pm 70)N_f^2 + 1.5N_f^3 \quad (estimate), \quad (5.11)$$

(the numbers between the parenthesis are the error estimates) whereas exactly [12]

$$\beta_3^{\overline{MS}} = 24633 - 6375N_f + 398.5N_f^2 + 1.5N_f^3 \quad (exact). \quad (5.12)$$

Despite the remarkable agreement, a note of warning must be said. The Padé approximants predictions, being constructed from the lower orders coefficients, cannot account for the effects of terms which are topologically new in higher orders of perturbation theory. This problem reveals itself in the existence of quartic Casimir effects in the calculation of  $\beta_3$  which appear for the first time at four-loops order and were omitted in (5.12).

This strategy of estimating higher order coefficients in perturbative series has been further extended to the anomalous mass dimension function, the Higgs decay rate and scalar/pseudoscalar QCD sum rules [87], and also to supersymmetric QCD [88].

It was soon recognised that Padé approximants would be specially important to make sense out of QCD series dominated by renormalons and thus necessarily asymptotic [36]. This important property of Padé approximants is not surprising

if we recall our explorations in Chapter 2. Also, the possible dominance of renormalons already in low orders would provide an explanation for the success of Padé approximants in predicting the next unknown coefficient in a QCD perturbative series.

Another very important feature of Padé approximants is that they may reduce the RS-dependence of the QCD perturbative calculations. This has been observed for the case of the polarised Bjorken sum rule [36, 37]. Moreover, in the large  $b$  limit diagonal Padé approximants become exactly renormalisation scale independent [38] and, as seen before, the leading- $b$  approximation is of vital importance in our approach to large-orders perturbative QCD. There have also been studies on the combination of the PMS with Padé approximants to determine  $\mu$  and  $c_2$  [89, 90].

It was suggested in one of the first papers in this area [36] that it could be fruitful to calculate Padé approximants to the Borel transform rather than to the original series. This has been pursued in [43], where it was applied to the study of the asymptotic series for the running coupling constant obtained from the static QCD potential modeled by the Richardson potential, and to other problems. It is found that the Padé approximants singularities cluster about the singularities of the Borel transform, reproducing poles exactly when these exist in a finite number. This approach, sometimes termed *Borel-Padé method*, has also been applied to the perturbative expansion of the QED effective action in a constant background electric field (which diverges factorially), and to the perturbative series of the Stark energy shift (which consists of a combination of fixed-sign and alternating-sign factorial behaviours) [91]. The Borel-Padé method is there recognised as a way of analytically continuing the Borel transform in the complex plane. Another approach aimed at the improvement of the Borel summations enlarges the domain of analyticity (which goes until the first Borel transform singularity) by doing a conformal mapping on the Borel transform variable [92]. A combination of conformal mapping and Padé approximants can be found in [93]. For a comprehensive set of references in the growing field of Padé approximants and related methods, see [47].

In this Chapter's practical studies, we shall be concerned with Padé approximants to the series (5.6). We will always be using the “normal” sequence of Padé approximants, which is the one generated by the continued fractions. It is sensible to use

this sequence of diagonal/off-diagonal Padé approximants because it is known to converge to the Borel sum when the series is Stieltjes, and also because diagonal Padé approximants reduce the renormalisation-scale dependence [38].

Explicitly, our Padé approximants will be obtained from the truncations of the continued fraction

$$R(a_0) = a_0 + X_2 a_0^3 \frac{K_0}{1 + \frac{K_1 a_0}{1 + \frac{K_2 a_0}{1 + \frac{K_3 a_0}{1 + \frac{K_4 a_0}{1 + \dots}}}}}, \quad (5.13)$$

with

$$K_0 = 1, \quad (5.14)$$

$$K_1 = -\frac{X_3^{(L)}}{X_2}, \quad (5.15)$$

$$K_2 = \frac{X_3^{(L)}}{X_2} - \frac{X_4^{(L)}}{X_3^{(L)}}, \quad (5.16)$$

$$K_3 = \frac{X_4^{(L)}}{X_3^{(L)}} - \frac{X_3^{(L)} X_4^{(L)} - X_2^{(L)} X_5^{(L)}}{X_3^{(L)2} - X_2^{(L)} X_4^{(L)}}, \quad (5.17)$$

$$\vdots \quad \vdots$$

#### 5.1.4 Numerical results for the Adler function $D(a)$

In Figure 5.1 we present the Adler function  $D(a)$  in the CORGI scheme resummed with the various methods described before, at high energy. We observe that both the Padé approximants and the truncated continued functions method converge to the Borel sum. The agreement is of eight digits for the Padé approximants, and seven digits for the truncated continued function method, at the highest order shown. We note that the divergence of the “naive perturbation theory” is clearly akin to an alternating-sign factorial series. The ambiguity in the Borel sum from

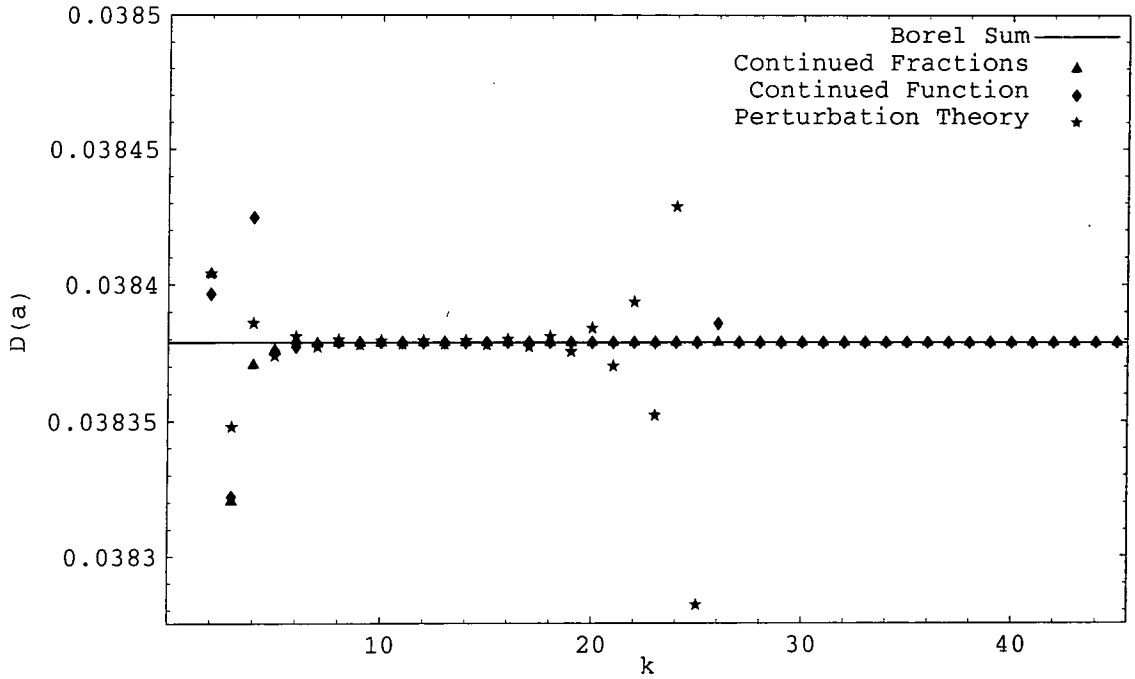


Figure 5.1: The Adler function  $D(a)$  in the CORGI scheme resummed with  $Q=91$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=200$  MeV, and  $N_f=5$  flavours.

the first  $IR$  renormalon ( $IR_2$ ) is extremely small at this energy ( $\approx 10^{-11}$ ). We have no explanation for the extraneous point at  $\mathcal{O}(a^{27})$  for the  $G(G\dots)$ , but even orders are part of the interpolated sequence and thus do not lead the convergence of the truncated continued functions method.

In Figure 5.2 we show the Adler function  $D(a)$  in the CORGI scheme resummed at an intermediate energy. The agreement between the Padé approximants and the Borel sum is never better than three digits up to the order shown, but most of the Padé approximants highest orders stay within the ambiguity of the first  $IR$  pole in the Borel sum (whose limits are shown by the dashed lines). The truncated continued functions have four digits fixed at the order shown, but the limit they are converging to is distinct from the Borel sum. In fact, the ambiguity from the first  $IR$  renormalon is big enough to engulf the best of the Padé approximants points, but not the truncated continued function method points. However, the difference between the Borel sum and the  $G(G\dots)$  is actually less than 1%.

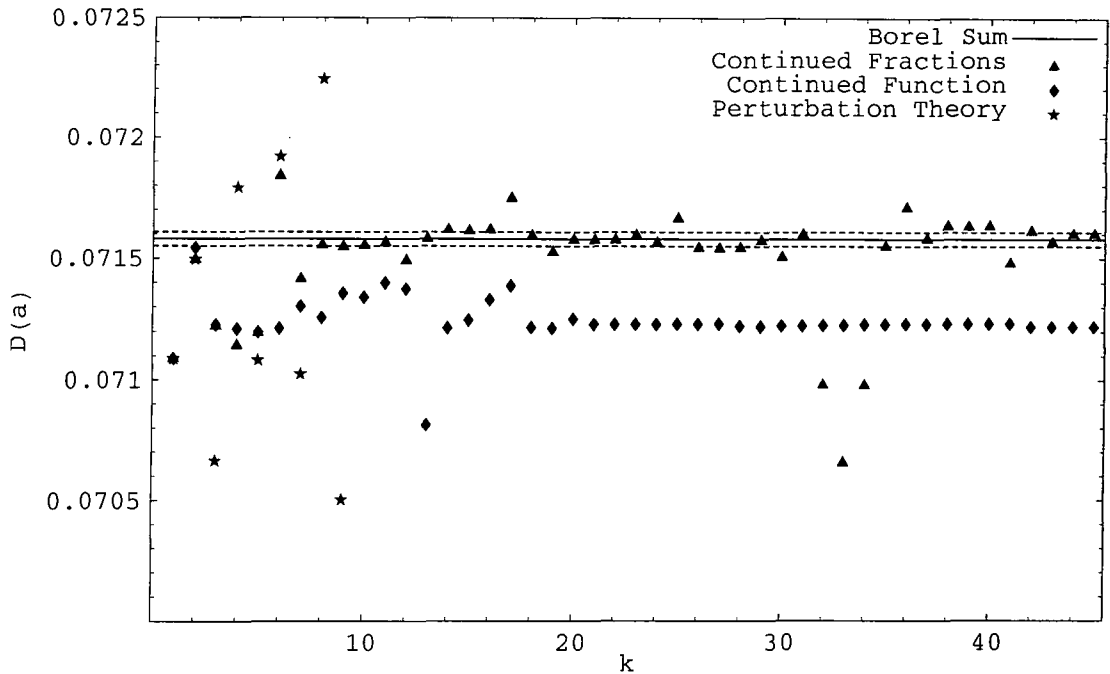


Figure 5.2: The Adler function  $D(a)$  in the CORGI scheme resummed with  $Q = 5$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 279$  MeV, and  $N_f = 4$  flavours.

We can see in Figure 5.3 that, at a low energy, the effect of the  $IR$  renormalons delays the convergence of the truncated continued function method until a very high order, and that of the Padé approximants until an even higher order. However, the predominance of the  $UV$  renormalons for this observable asserts itself, and we have convergence even at such a low energy. The difference between the Borel sum and the truncated continued function method limit is only of 5.8%. The difference between the Padé approximants and the Borel sum, at the order shown, is again better (3.2%), and it tends to diminish asymptotically, provided one ignores two subsequences which show at even orders (the Padé approximants points have a period of six). Both the truncated continued function method and the Padé approximants, at the order shown, are within the ambiguity of the Borel sum estimated from the first  $IR$  renormalon, which corresponds to a power correction  $\left(\frac{\tilde{\Lambda}_{\overline{MS}}}{Q}\right)^4$ .

A comment is in order concerning the convergence of the truncated continued function method. We suggested in section 4.3 that the convergence of at least the

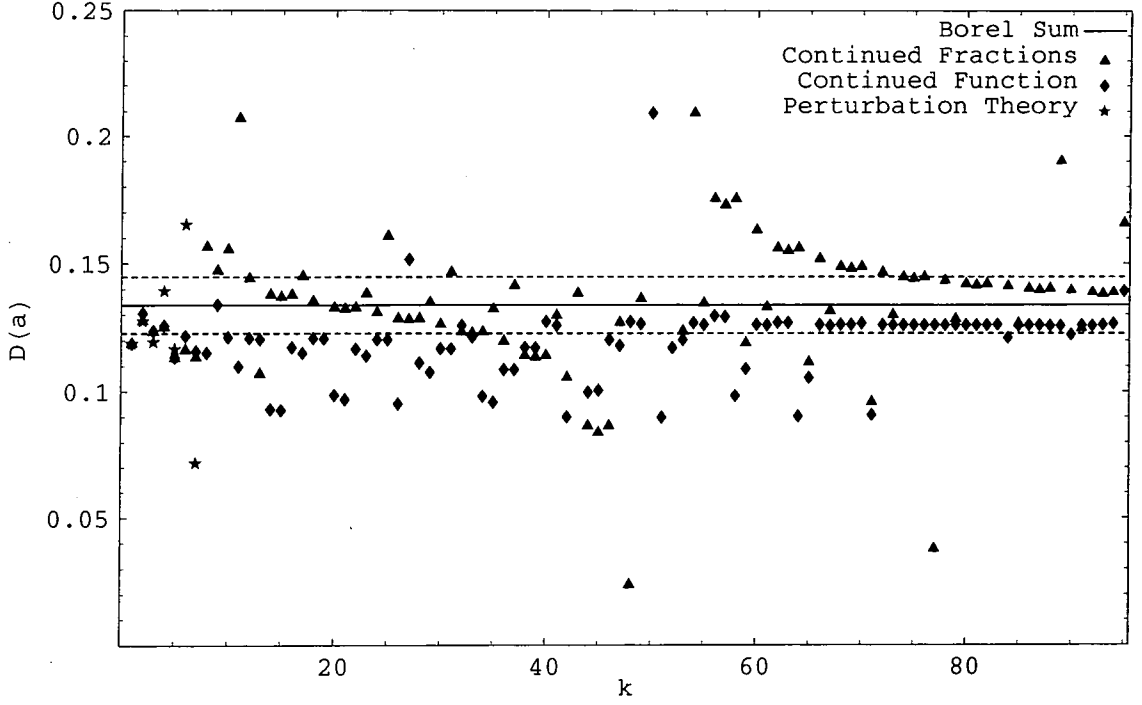


Figure 5.3: The Adler function  $D(a)$  in the CORGI scheme resummed with  $Q=1.5$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=320$  MeV, and  $N_f=3$  flavours.

“natural” sub-sequence would be guaranteed if the growth of the  $\rho_0^{(k)}$ ’s was dominated by a  $UV$  renormalon, and thus linear. It is indeed true that the Adler function  $D(a)$  perturbation theory coefficients  $d_k^{(k)}$  in the  $V$ -scheme behave, in first approximation, like a  $UV$  renormalon [60] and, as a consequence, the CORGI series coefficients  $X_k^{(L)}$  themselves behave like an alternating-sign factorial. Then, if we consider only the leading- $b$  coefficients in the  $V$ -scheme, “switching off” the contribution from the exactly known  $d_1^{\overline{MS}}, d_2^{\overline{MS}}, c_2^{\overline{MS}}$ , the growth of the  $\rho_0^{(k)}$ ’s and  $\rho_2^{(k)}$ ’s conforms to expectations. However, the exact  $X_2$  has a different sign and size from  $X_2^{(L)}$ , and its introduction alone is enough to destroy the linear behaviour of the  $\rho_0^{(k)}$ ’s studied before. In fact, the  $\rho_0^{(k)}$ ’s do not have a recognisable growing pattern at all, and can take negative values. Nevertheless, the truncated continued function method still converges when applied to the Adler function  $D(a)$  (and at high energy seems to do so to the Borel sum), as we can see in Figures 5.1, 5.2, and 5.3. This can be understood, intuitively, as meaning that the structure of the truncated con-

tinued functions tames powerfully any wild behaviour of the coefficients, and indeed only a very special growth of the coefficients would lead to a sequence of divergent truncated continued functions. This is not the case for the Adler function  $D(a)$ , where the  $\rho_0^{(k)}$ 's and  $\rho_2^{(k)}$ 's seem to remain, asymptotically, within a limited range, quite moderate in size.

The asymptotic behaviour of the Padé approximants  $K_j$ 's is not destroyed by the replacement of the exact  $X_2$  for the leading- $b$   $X_2^{(L)}$ . The only effect here is to delay by a few orders the onset of the sign pattern already advanced in Table 2.3. It must also be noted that whereas the poles and zeros all sit on the real axis when we consider only renormalons (this is not surprising if we recall that the Adler  $D(a)$  renormalons can be expressed as a Hamburger function), they become generally complex when the exact coefficients are included. The majority of the singularities lie on the negative half of the complex plane, but sequences of poles exist on the positive side. Therefore, there may be exceptionally bad results for some Padé approximants at given orders which have poles close to the value of the coupling constant.

## 5.2 Deep inelastic scattering (DIS) sum rules

In this section, we shall be interested in the three deep inelastic scattering (DIS) sum rules: the polarised and unpolarised Bjorken sum rules, and the Gross-Llewellyn Smith sum rule. We shall proceed along the same lines of the previous section, and redundant remarks will be avoided.

### 5.2.1 Polarised Bjorken sum rule (PBjSR)

The polarised Bjorken sum rule (PBjSR) is defined as:

$$K_{PBj} \equiv \int_0^1 g_1^{ep-en}(x, Q^2) dx = \frac{1}{3} \left| \frac{g_A}{g_V} \right| \left( 1 - \frac{3}{4} C_F K(a) \right), \quad (5.18)$$

where  $g_V$  and  $g_A$  denote the nucleon vector and axial vector couplings, respectively. The quantity we shall be interested in here is  $K(a)$ , the perturbative corrections to the zeroth order parton model sum rule,



$$K(a) = a + K_1 a^2 + K_2 a^3 + \dots K_k a^{k+1} + \dots \quad (5.19)$$

The exact coefficients  $K_1$  and  $K_2$  have been computed [94, 95]. In the  $\overline{MS}$  scheme at the physical scale  $\mu=Q$ , they are

$$K_1 = -\frac{1}{3}N_f + \frac{23}{12}N - \frac{7}{8}C_F, \quad (5.20)$$

$$K_2 = \frac{115}{648}N_f^2 + N\left(-\frac{3535}{1296} - \frac{1}{2}\zeta_3 + \frac{5}{9}\zeta_5\right)N_f + C_F\left(\frac{133}{864} + \frac{5}{18}\zeta_3\right)N_f \\ + N^2\left(\frac{5437}{648} - \frac{55}{18}\zeta_5\right) + N\left(-\frac{1241}{432} + \frac{11}{9}\zeta_3\right)C_F + \frac{1}{32}C_F^2. \quad (5.21)$$

### 5.2.2 Gross-Llewellyn Smith sum rule (GLSSR)

The Gross-Llewellyn Smith sum rule (GLSSR) is defined as

$$K_{GLS} \equiv \frac{1}{6} \int_0^1 F_1^{\bar{\nu}p+\nu p}(x, Q^2) dx = 1 - \frac{3}{4}C_F K(a) + \tilde{K}(a). \quad (5.22)$$

The perturbative corrections  $K(a)$  are the same as for PBjSR. The additional corrections  $\tilde{K}(a)$  are of the “light-by-light” type and enter once again only at  $\mathcal{O}(a^3)$  (as in the case of the Adler function), and are thus expected to be small.

### 5.2.3 Large-order coefficients and Borel plane singularities of the PBjSR and GLSSR $K(a)$

The generating function for the leading- $b$  coefficients for the PBjSR and GLSSR  $K(a)$  in the V-scheme is [59]

$$K_k^{(k)} = \frac{1}{3} \left(\frac{1}{2}\right)^k \frac{d^k}{dx^k} \frac{3+x}{(1-x^2)(1-\frac{x^2}{4})} \Big|_{x=0}. \quad (5.23)$$

The Borel transform turns out to be [60]

$$B[K](z) = \frac{8}{9} \frac{1}{2+bz} - \frac{2}{9} \frac{1}{4+bz} + \frac{16}{9} \frac{2}{2-bz} - \frac{10}{9} \frac{1}{4-bz}. \quad (5.24)$$

Thus the Borel plane singularity structure of the PBjSR (and GLSSR) consists of four simple poles:  $UV_1$ ,  $UV_2$ ,  $IR_1$ , and  $IR_2$ , respectively. The subleading (of  $\mathcal{O}(1/N_f)$ )

corrections were here ignored. The actual large-order coefficients  $K_k^{(k)}$  are deduced to be

$$K_k^{(k)} = \frac{4}{9} \left(-\frac{1}{2}\right)^k k! + \frac{8}{9} \left(\frac{1}{2}\right)^k k! - \frac{1}{18} \left(-\frac{1}{4}\right)^k k! - \frac{5}{18} \left(\frac{1}{4}\right)^k k!. \quad (5.25)$$

Even in the lowest orders,  $K_k^{(k)}$  is numerically dominated by the two leading singularities,  $UV_1 + IR_1$  [60]. The Borel sum can now be trivially calculated,

$$\begin{aligned} K(x) = & -\frac{4}{9} z_1 e^{\frac{z_1}{x}} Ei\left(-\frac{z_1}{x}\right) + \frac{1}{18} z_2 e^{\frac{z_2}{x}} Ei\left(-\frac{z_2}{x}\right) \\ & + \frac{8}{9} z_1 e^{-\frac{z_1}{x}} Ei\left(\frac{z_1}{x}\right) - \frac{5}{18} z_2 e^{-\frac{z_2}{x}} Ei\left(\frac{z_2}{x}\right). \end{aligned} \quad (5.26)$$

The Borel sum in either the CORGI scheme or the V-scheme can be obtained from the equation above using the appropriate argument “ $x$ ”.

The series (5.25) (which is the same as (2.58)) is not a Stieltjes series, but has the Hamburger function representation (2.73). The Hamburger function representation, physically speaking, is the momentum distribution function. This fact was used in [63] to study how the first few exact perturbative coefficients can be used to impose bounds on the all-orders momentum distribution function. This was done making use of the fact that the all-orders momentum distribution function in the leading- $b$  approximation is known, and that Padé approximants to it can be calculated with the first few exact coefficients. Assuming that the uncertainty in the Padé approximants is of the same order as the next unknown term in the power series, one can calculate an upper and a lower bound for the momentum distribution function from the exact coefficients. It is found that the Adler  $D(a)$  all-orders momentum distribution function stays within the bounds (the same is expected for the PBjSR and GLSSR  $K(a)$ ). It must be noted that requiring the positivity of the momentum distribution function (a theoretically unjustified assumption) is the same as having it defining a Hamburger function. We already saw we have a Hamburger function for the Adler function  $D(a)$  (see (3.38)), and this is also the case for the PBjSR (or GLSSR)  $K(a)$  in the leading- $b$  approximation and in the V-scheme (see (2.73)).

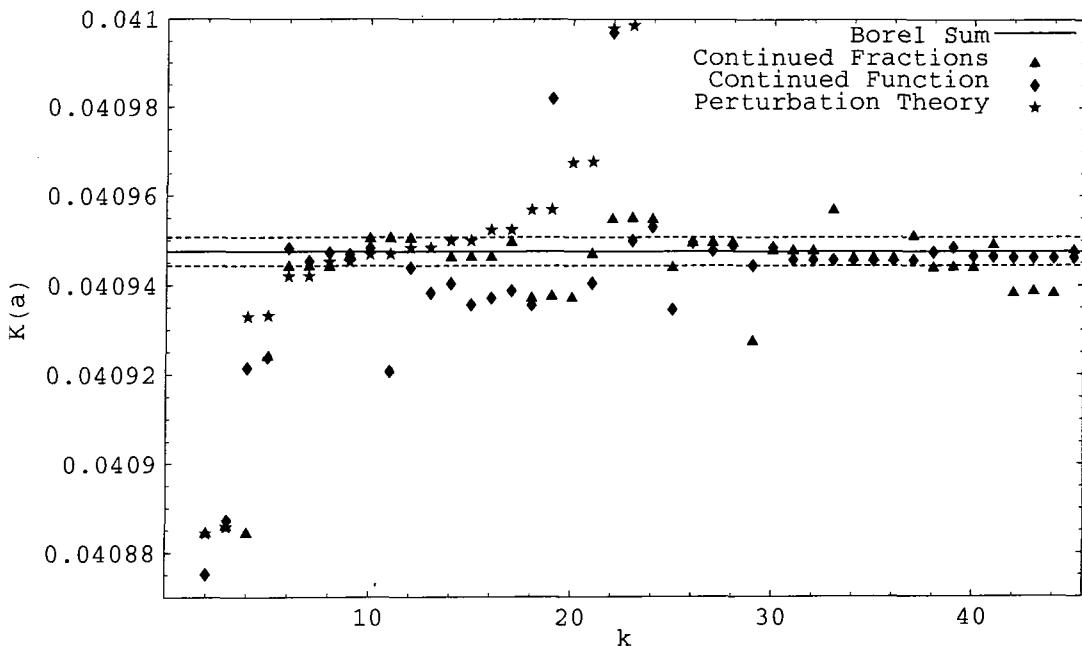


Figure 5.4: The PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed with  $Q=91$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=200$  MeV, and  $N_f=5$  flavours.

#### 5.2.4 Numerical results for the PBjSR and GLSSR $K(a)$

We show the PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed at an high energy in Figure 5.4. We first note that the divergence of the “naive perturbation theory” series shows the presence of the leading  $UV_1 + IR_1$  singularities by diverging exponentially, with the points pairing up. The oscillations of the truncated continued function method, already at the order shown, are within the ambiguity of the Borel sum given by the first  $IR$  renormalon ( $IR_1$ ), which is of the same magnitude as  $\left(\frac{\tilde{\Lambda}_{\overline{MS}}}{Q}\right)^2$  and is marked by the dashed lines. The agreement is of four digits. The Padé approximants oscillate in and out of the Borel sum ambiguity area, sometimes agreeing with the Borel sum to three or four digits, but never really settling. It must be noted that the sign pattern for the continued fraction  $K_j$ ’s is  $++--++--\dots$ , and that the poles and zeros divide themselves evenly on both sides of the plane. This conforms to expectations from Table 2.2 and Table 2.3 in subsection 2.3.3, and may explain why the Padé approximants are so unstable in this case. In fact, it is

close to the origin that the Padé approximants poles are more dense, and it is at high energy that the coupling constant is small enough to be in the dangerous zone. As it happened in the case of the Adler function  $D(a)$ , the  $\rho_0^{(k)}$ 's and  $\rho_2^{(k)}$ 's do not have a recognisable behaviour. They stay within a bounded range of values at high orders, and that seems to be enough to assure the stability of the  $G(G\dots)$  results.

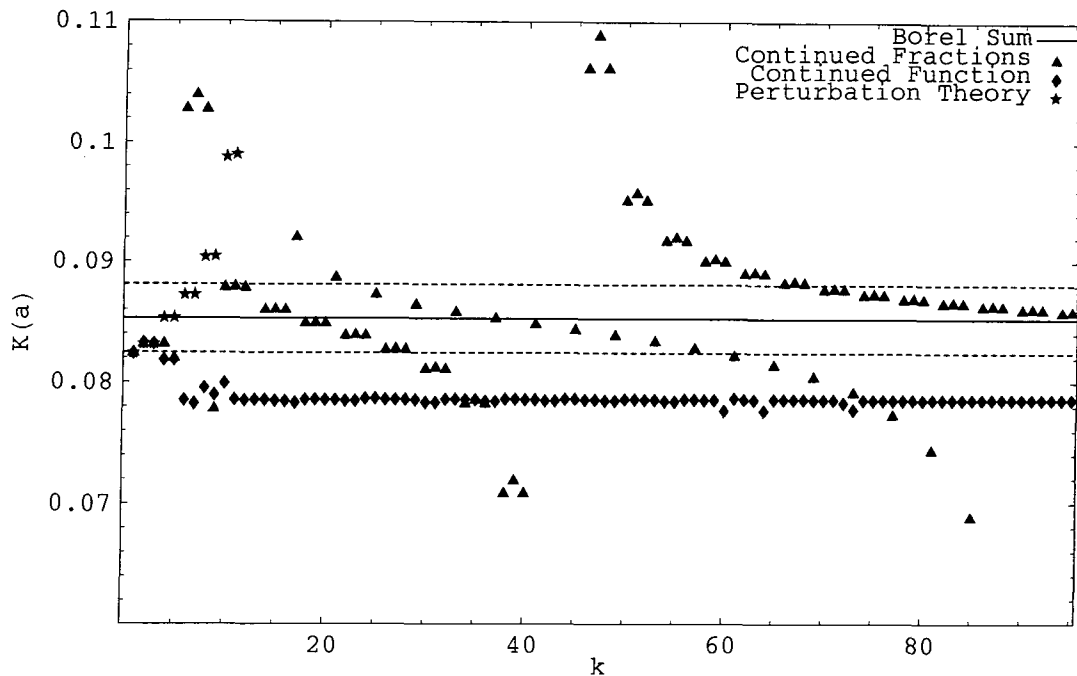


Figure 5.5: The PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed with  $Q=5$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=279$  MeV, and  $N_f=4$  flavours.

We show the PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed at an intermediate energy in Figure 5.5. The divergence of the “naive perturbation theory” series is even more immediate and sharp. The Padé approximants only converge at a very high order. They reach the Borel sum ambiguity at  $\mathcal{O}(a^{67})$ , if we ignore a subsequence which shows up once in every four Padé approximants. The truncated continued function method points converge to a different limit, actually outside the reach of the Borel sum ambiguity bars. It must be noted that their convergence is much faster than the one of the Padé approximants.

We show the PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed at a

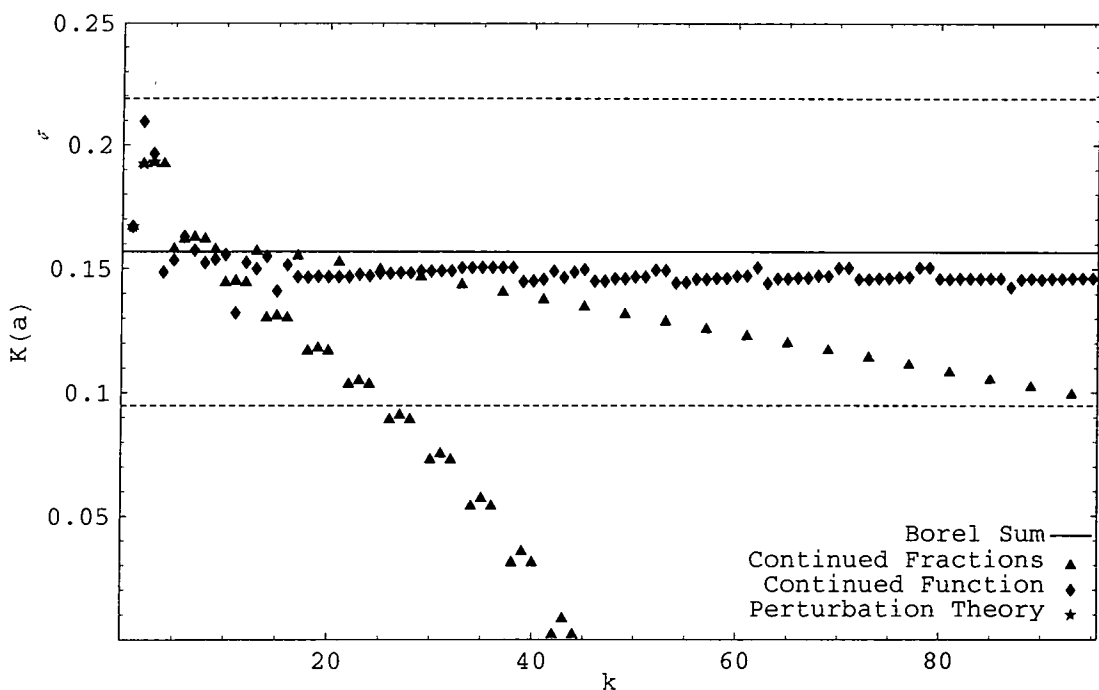


Figure 5.6: The PBjSR (and GLSSR)  $K(a)$  in the CORGI scheme resummed with  $Q = 1.5$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 320$  MeV, and  $N_f = 3$  flavours.

low energy in Figure 5.6. The most striking feature is that the Padé approximants are hopelessly divergent. Both sub-sequences shoot off to highly unlikely negative values. On the other hand, the truncated continued function method has two digits fixed at the order shown, and within the Borel sum ambiguity. Thus, the truncated continued function method does show its superiority in this case.

### 5.3 Unpolarised Bjorken sum rule (BjSR)

The unpolarised Bjorken sum rule (BjSR) is defined as

$$U_{Bj} \equiv \int_0^1 F_1^{\bar{\nu}p-\nu p}(x, Q^2) dx = 1 - \frac{1}{2} C_F U(a). \quad (5.27)$$

Similarly to the polarised case,  $U(a)$  is the quantity that consists of perturbative corrections to the zeroth order parton model, with coefficients that we shall denote by  $U_k$ ,

$$U(a) = a + U_1 a^2 + U_2 a^3 + \dots U_k a^{k+1} + \dots \quad (5.28)$$

The known coefficients are again the first two [96, 97]. In the  $\overline{MS}$  scheme with renormalisation scale  $\mu = Q$  they are

$$U_1 = -\frac{4}{9}N_f + \frac{91}{36}N - \frac{11}{8}C_F, \quad (5.29)$$

$$\begin{aligned} U_2 = & \frac{155}{648}N_f^2 + N\left(-\frac{4235}{1296} + \frac{7}{6}\zeta_3 - \frac{5}{3}\zeta_5\right)N_f + C_F\left(\frac{335}{288} - \frac{1}{6}\zeta_3\right)N_f \\ & + N^2\left(\frac{8285}{648} + 5\zeta_3 - 10\zeta_5\right) + N\left(-\frac{2731}{144} - \frac{91}{3}\zeta_3 + \frac{95}{2}\zeta_5\right)C_F \\ & + \left(\frac{313}{32} + \frac{47}{2}\zeta_3 - 35\zeta_5\right)C_F^2. \end{aligned} \quad (5.30)$$

The generating function for the unpolarised Bjorken sum rule in the V-scheme turns out to be [98]

$$U_k^{(k)} = \left(\frac{1}{2}\right)^k \frac{d^k}{dx^k} \frac{1}{(1-x)(1-\frac{x^2}{4})} \Big|_{x=0}. \quad (5.31)$$

The Borel transform is then deduced as

$$B[U](z) = \frac{8}{3} \frac{1}{2-bz} - \frac{2}{4-bz} + \frac{2}{3} \frac{1}{4+bz}. \quad (5.32)$$

It is to be noted that this corresponds to  $IR_1, IR_2$ , and  $UV_2$ , respectively. This is, thus, an observable dominated by a  $IR$  singularity. The large-orders coefficients can then be written as

$$U_k^{(k)} = \frac{4}{3} \left(\frac{1}{2}\right)^k k! - \frac{1}{2} \left(\frac{1}{4}\right)^k k! + \frac{1}{6} \left(-\frac{1}{4}\right)^k k!, \quad (5.33)$$

and the Borel sum is

$$U(x) = \frac{1}{6} z_2 e^{\frac{z_2}{x}} Ei\left(-\frac{z_2}{x}\right) + \frac{4}{3} z_1 e^{-\frac{z_1}{x}} Ei\left(\frac{z_1}{x}\right) - \frac{1}{2} z_2 e^{-\frac{z_2}{x}} Ei\left(\frac{z_2}{x}\right). \quad (5.34)$$

This Borel sum could also be written as a Hamburger function.

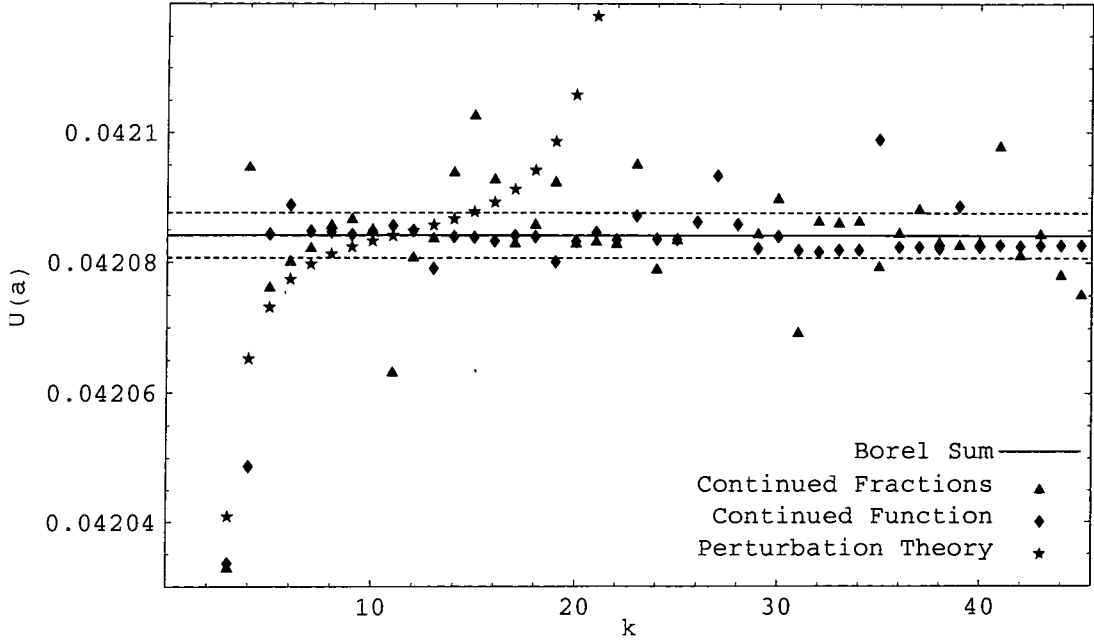


Figure 5.7: The Unpolarised Bjorken Sum Rule  $U(a)$  in the CORGI scheme resummed with  $Q=91$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=200$  MeV, and  $N_f=5$  flavours.

### 5.3.1 Numerical results for the BjSR $U(a)$

The resummations of the unpolarised Bjorken sum rule  $U(a)$  in the CORGI scheme at an high energy are presented in Figure 5.7. The “naive” perturbation theory series diverges early and exponentially, a fact which reflects the predominance of a  $IR$  singularity. Most of the truncated continued function method points stay within the ambiguity of the Borel sum estimated by this observable’s first singularity on the domain of integration,  $IR_1$  (the ambiguity limits are indicated by the dashed lines), already at the order shown. The Padé approximants are more erratic, and stay so at orders higher than the ones shown. It must be noticed that the sign pattern of the continued fractions  $K_j$ ’s (+ + - - - + + - - - ...) is symmetrical of the one of the Adler function  $K_j$ ’s. (This fact reminds us that the combination of renormalons here discussed is symmetrical in the Borel plane to the three leading singularities in the Adler function  $D(a)$ .) Accordingly, the Padé approximants have more poles (and zeros) on the right-hand side of the real axis than on the negative side. The truncated continued function method  $\rho_0^{(k)}$ ’s and  $\rho_2^{(k)}$ ’s have an even more

irregular behaviour than they had for the two cases previously studied. About half of the  $\rho_0^{(k)}$ 's have a negative sign. This does not seem to disturb the convergence properties too badly.

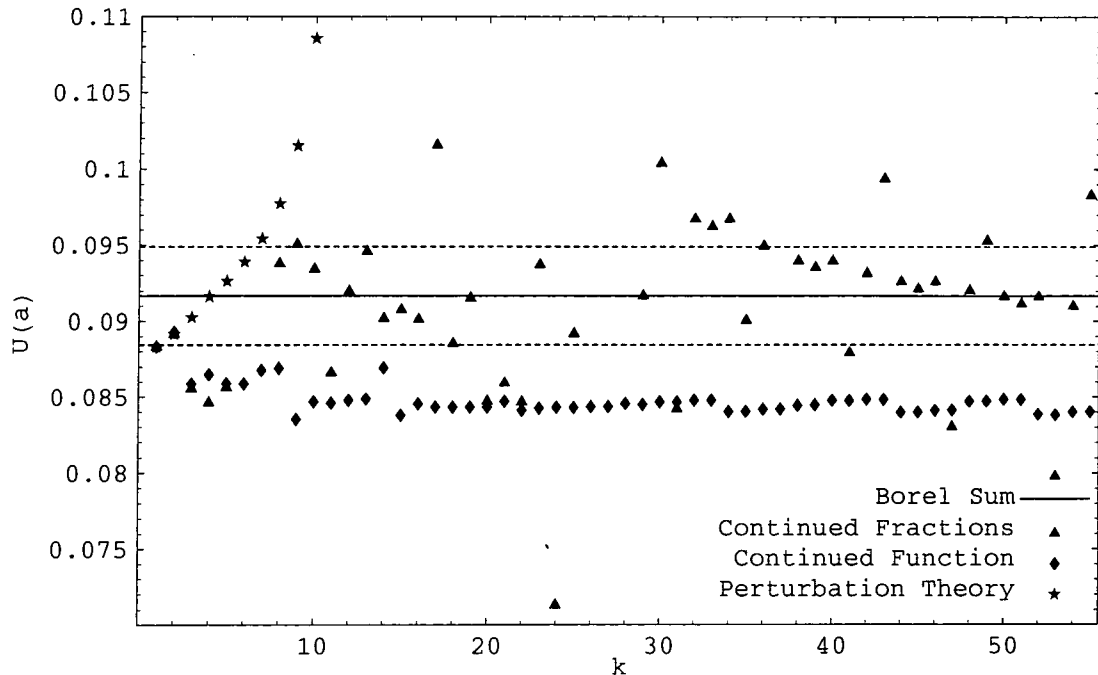


Figure 5.8: The Unpolarised Bjorken Sum Rule  $U(a)$  in the CORGI scheme resummed with  $Q=5$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=279$  MeV, and  $N_f=4$  flavours.

The resummations of the unpolarised Bjorken sum rule  $U(a)$  in the CORGI scheme at an intermediate energy are presented in Figure 5.8. The Padé approximants converge within the Borel sum ambiguity (shown by the dashed lines), but rather slowly, and only provided we ignore two points out of every six. The truncated continued function method converges much faster, and to a more definite limit than the Padé approximants. However, such limit is not within the Borel sum ambiguity.

The resummations of the unpolarised Bjorken sum rule  $U(a)$  in the CORGI scheme at a low energy are shown in Figure 5.9. The Padé approximants do not converge at all. Once again, the truncated continued function method points converge to a definite limit within the Borel sum ambiguity (but which only agrees with the Borel sum to a digit).



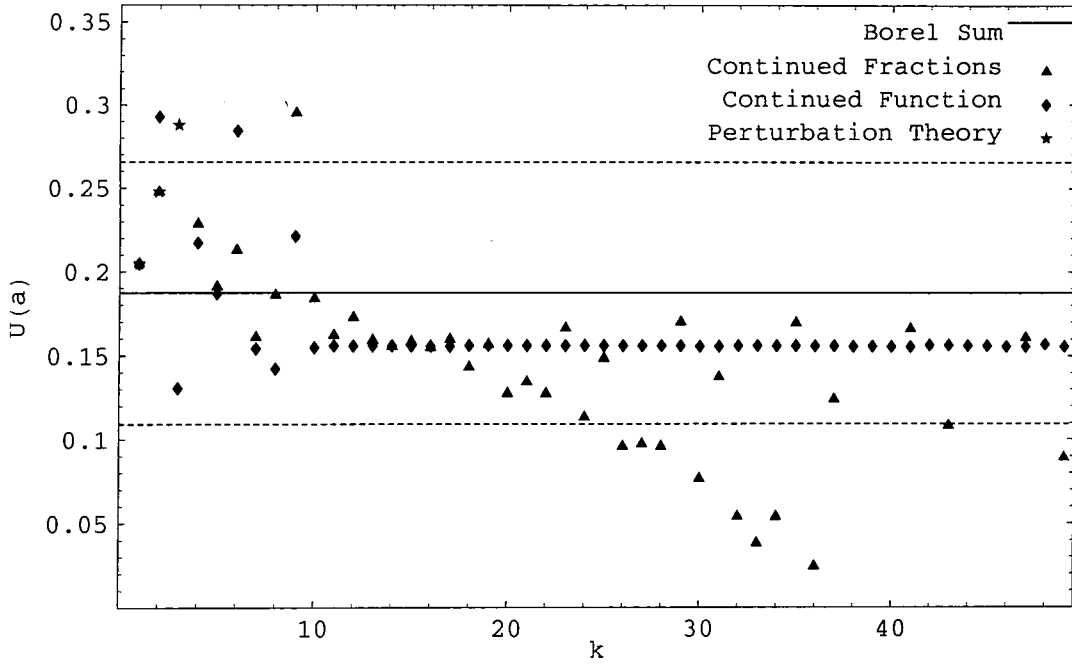


Figure 5.9: The Unpolarised Bjorken Sum Rule  $U(a)$  in the CORGI scheme resummed with  $Q=1.5$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=320$  MeV, and  $N_f=3$  flavours.

## 5.4 Summary and conclusions

The truncated continued function method was studied as a resummation procedure for large-orders QCD. From the three observables examined, it is possible to conclude that it converges asymptotically to a different limit than the Padé approximants limit/Borel sum value. However, it was shown that it works where Padé approximants fail (namely at lower energies, and specially when *IR* renormalons are predominant), and that it reaches a stable value much faster than the Padé approximants. This stability is very striking, but it is not obvious what “limit” is being selected. It is not, evidently, the Cauchy Principal Value of the Borel sum.

## Chapter 6

# Reformulated perturbative QCD for contour-improved observables

In this Chapter, we shall be concerned with the application to the calculation of experimentally measurable physical quantities of the reformulation of perturbative QCD described in Chapter 4. These physical quantities shall be the  $\tau$  decay ratio  $R_\tau$ , and the R-ratio for  $e^+e^-$  annihilation into hadrons. It turns out that both of these quantities can be expressed as contour integrals of  $D(se^{i\theta})$ , where  $D(-s)$  is the euclidean Adler function, and  $s=Q^2$  is the squared center-of-mass momentum. For this reason, we shall refer to these Minskowskian quantities represented in this way as *contour-improved* observables. In many aspects, the work presented in this Chapter will be very similar to the one in the previous Chapter. For instance, the resummation methods to be compared with reformulated perturbative QCD (or truncated continued function method) will be the same as in the previous Chapter.

In this Chapter, we will start by defining the variables, and we will proceed by discussing their contour integral representation. How to perform calculations of CORGI “naive” perturbation theory, Borel sums, Padé approximants, and how to use the truncated continued function method in the contour integral representation will then be described shortly. We will compare the several resummation methods with the experimental values available for these observables, an exercise which will allow us to find an experimentally fitted value for the fundamental QCD parameter  $\tilde{\Lambda}$ , and, enabling us to make comparisons with a large body of references in the literature, we will then determine  $\alpha_s(Q/\tilde{\Lambda})$ . This is, thus, the ultimate test for our

resummation method.

## 6.1 The $\tau$ decay ratio $R_\tau$

The semi-leptonic branching ratio of the  $\tau$  lepton decay to hadrons ( $\tau \rightarrow \nu_\tau, \text{hadrons}, n\gamma$ ) is an inclusive quantity and as such a rigorous theoretical calculation is possible. Thus, the decays of the  $\tau$  will enable us to compare perturbative QCD with experiment, and this in a low energy region where power corrections could be significant (the  $\tau$  lepton has a mass of only 1.777 GeV).

In order to factor out the weak interactions, we normalise the  $\tau$  decay hadronic width as

$$R_\tau \equiv \frac{\Gamma(\tau \rightarrow \nu_\tau, \text{hadrons}, n\gamma)}{\Gamma(\tau \rightarrow \nu_\tau e^- \bar{\nu}_e)}. \quad (6.1)$$

Following [99], we decompose  $R_\tau$  as

$$R_\tau = N(|V_{ud}|^2 + |V_{us}|^2) S_{EW} \left(1 + \frac{3}{4} C_F \tilde{R}_\tau + \delta_{EW} + \delta_{PC}\right), \quad (6.2)$$

where  $V_{ud}$  and  $V_{us}$  are matrix entries of the Cabibo-Kobayashi-Maskawa matrix,  $S_{EW}$  and  $\delta_{EW}$  are electro-weak radiative corrections,  $\delta_{PC}$  stands for the power corrections, and  $\tilde{R}_\tau$  is the quantity which is expanded in a power series in the coupling. In the following, we will restrict ourselves to non-strange decays of the  $\tau$ , removing events containing strange quarks from the data, and thus one must set  $V_{us} = 0$ . Following [100], we will use the experimental value  $V_{ud} = 0.9752 \pm 0.0007$ . The electroweak correction  $S_{EW}$  is logarithmic and given by<sup>1</sup>[101]

$$S_{EW} = \left(\frac{a_{QED}(m_b^2)}{a_{QED}(m_\tau^2)}\right)^{9/19} \left(\frac{a_{QED}(M_W^2)}{a_{QED}(m_b^2)}\right)^{9/20} \left(\frac{a_{QED}(M_Z^2)}{a_{QED}(M_W^2)}\right)^{36/17} \simeq 1.0194 \pm 0.0040, \quad (6.3)$$

where  $a_{QED}(Q^2)$  is the QED running coupling constant ( $a_{QED} = \frac{\alpha}{\pi}$ ), and whilst the non-logarithmic correction  $\delta_{EW}$  is given by [102]

$$\delta_{EW} = \frac{5}{12} a_{QED}(m_\tau^2) \simeq 0.001. \quad (6.4)$$

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<sup>1</sup> $\alpha(m_b^2) = \frac{1}{132.05}; \alpha(M_W^2) = \frac{1}{127.97}; \alpha(M_Z^2) = \frac{1}{127.93}.$

The contribution from truly nonperturbative terms can be estimated using sum rules [103]. The estimated value is [99, 104, 105]

$$\delta_{PC} = -0.014 \pm 0.005. \quad (6.5)$$

### 6.1.1 The experimental $R_\tau$

The experimental value for the  $R_\tau$ , with the contribution from strange quarks excluded, is actually obtained from the measurements of the branching ratios  $B_e$  and  $B_\mu$ ,

$$R_\tau^{exp.} \equiv \frac{1 - B_e - B_\mu}{B_e}, \quad (6.6)$$

where  $B_\ell \equiv \Gamma(\tau \rightarrow \nu_\tau \ell \bar{\nu}_\ell, \gamma) / \Gamma_T$  ( $\Gamma_T$  is the total decay rate). Since [100]

$$R_\tau^{exp.} = 3.492 \pm 0.016, \quad (6.7)$$

one has

$$\tilde{R}_\tau^{exp.} = 0.214 \pm 0.017. \quad (6.8)$$

## 6.2 The R-ratio decay rate

Hadronic production through the mechanism  $e^+e^- \rightarrow (Z \text{ and/or } \gamma) \rightarrow \text{hadrons}$  is fully inclusive, and as such it is also a good test of perturbative QCD. The R-ratio for  $e^+e^-$  annihilation into hadrons is defined as

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons}, \gamma)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-, \gamma)}. \quad (6.9)$$

This quantity can be calculated in perturbation theory as

$$R(s) = N \sum_f Q_f^2 \left( 1 + \frac{3}{4} C_F R \right) + \left( \sum_f Q_f \right)^2 \tilde{R}, \quad (6.10)$$

where  $s \equiv Q^2$  is the physical time-like squared momentum transfer ( $\sqrt{s} = Q$  is the  $e^+e^-$  center of mass energy).  $R$  is the quantity which is perturbative expanded and

on which we shall concentrate.  $\tilde{R}$  only enters at  $\mathcal{O}(a^3)$  and is due to the existence of “light-by-light” diagrams.

### 6.2.1 The experimental R-ratio at the $Z^0$ energy

At the energy  $Q = m_{Z^0} (= 91.2 \text{ GeV})$ , there is an enormous amount of data. The hadronic decay width of the  $Z^0$  boson has thus been accurately measured as [106]

$$R_{Z^0} \equiv \frac{\Gamma(Z^0 \rightarrow \text{hadrons})}{\Gamma(Z^0 \rightarrow \text{leptons})} = 20.768 \pm 0.0024. \quad (6.11)$$

Since the theoretical prediction is

$$R_{Z^0}^{th} = 19.934(1 + 1.045R), \quad (6.12)$$

one can thus expect

$$R^{exp.} = 0.0400 \pm 0.0001. \quad (6.13)$$

## 6.3 The contour integral representation of Minkowski observables

In this section, we shall see how both the  $R_\tau$  and the R-ratio can be conveniently written as contour integrals.

### 6.3.1 The $R_\tau$ as a contour integral

One can define, analogously to the vectorial current (3.2), an axial-vector current

$$A^\mu \equiv \frac{1}{2} (: \bar{u} \gamma^\mu \gamma_5 u : - : \bar{d} \gamma^\mu \gamma_5 d :), \quad (6.14)$$

and a two-point correlation function for the axial-vector currents

$$\Pi_A^{\mu\nu}(q) \equiv i \int e^{iq \cdot x} \langle 0 | T \{ A^\mu(x) A^\nu(0)^\dagger \} | 0 \rangle d^4x. \quad (6.15)$$

Then, one has the Lorentz decompositions

$$\Pi_{V,A}^{\mu\nu}(q) = -(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi_{V,A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{V,A}^{(0)}(q^2), \quad (6.16)$$

where the superscript  $J=1,0$  denotes the angular momentum in the hadronic rest frame.

With these definitions in hand, the semihadronic decay rate of the  $\tau$  ( $R_\tau$ ) can be written as an integral over the invariant mass  $s=-q^2$  of the final state hadrons [99]:

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[ \left(1 + \frac{2s}{m_\tau^2}\right) \text{Im} \left\{ \Pi^{(1)}(s+i\epsilon) \right\} + \text{Im} \left\{ \Pi^{(0)}(s+i\epsilon) \right\} \right] \quad (6.17)$$

(the factors in curly brackets are due to the phase space integrations), where

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left( \Pi_{ud,V}^{(J)}(s) + \Pi_{ud,A}^{(J)}(s) \right). \quad (6.18)$$

Since we do not know how to account for the nonperturbative effects of QCD that bind quarks into hadrons, the integrand of (6.17) is not fully known. However, since the correlators are analytic functions of  $s$  with a cut along the positive real  $s$ -axis, one can re-express (6.17) as a contour integral running counter-clockwise from  $s=m_\tau^2+i\epsilon$  to  $s=m_\tau^2-i\epsilon$ ,

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[ \left(1 + \frac{2s}{m_\tau^2}\right) \Pi^{(1)}(s) + \Pi^{(0)}(s) \right]. \quad (6.19)$$

The advantage of this new expression is that knowledge of the correlators in the nonperturbative region  $|s| \ll m_\tau^2$  is no longer required. Then, as the combination  $\Pi^{(0+1)}(q^2) \equiv \Pi^{(0)}(q^2) + \Pi^{(1)}(q^2)$  must have a smoother limit than  $\Pi^{(0)}(q)$ ,  $\Pi^{(1)}(q)$  when  $q \rightarrow 0$  (see (6.16)), we rewrite (6.19) as

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[ \left(1 + \frac{2s}{m_\tau^2}\right) \Pi^{(0+1)}(s) - \frac{2s}{m_\tau^2} \Pi^{(0)}(s) \right]. \quad (6.20)$$

Now, since on the limit of massless quarks the term  $s\Pi^{(0)}(s)$  can be dropped [99], and recalling the definition of the Adler function (3.3), it results with integration by parts that

$$\tilde{R}_\tau = -12\pi i \oint_{|s|=m_\tau^2} \frac{ds}{s} \left( \frac{1}{2} - \frac{2}{m_\tau^2} + \frac{s^3}{m_\tau^6} - \frac{s^4}{2m_\tau^8} \right) D(s), \quad (6.21)$$

for the perturbative part alone, where  $D(s)$  is the Adler function  $D$ .

### 6.3.2 Power corrections to the $R_\tau$

With respect to the nonperturbative corrections  $\delta_{PC}$ , it must be noted that, given such a low energy as 1.777 GeV, it would not be surprising if power corrections were important. The form of the power corrections is suggested by the Operator Product Expansion. For scalar correlators, the operator product expansion takes the form [103]

$$\Pi^{(J)}(s) = \sum_{d=0,2,4,\dots} \frac{1}{(-s)^{d/2}} \sum_{\dim(\mathcal{O}_i)=d} C^{(J)}(s, \mu) \langle 0 | \mathcal{O}_i(\mu) | 0 \rangle, \quad (6.22)$$

where the inner sum is over local gauge invariant scalar operators, and  $\mu$  is an arbitrary factorization scale separating the nonperturbative effects which are related to the vacuum matrix elements ( $\langle 0 | \mathcal{O}_i(\mu) | 0 \rangle$ ), from the short-distance effects which are related to the Wilson coefficients  $C^{(J)}(s, \mu)$ . The operator of dimension  $d=0$  is the trivial unit operator. Why the operators of dimension  $d=2$  are absent was already discussed in sub-section 3.3.1. From these considerations, it follows that the power corrections to the  $R_\tau$  are expected to be of the form

$$\delta_{PC} = a \frac{\overline{m}^2}{m_\tau^2} + b \sum_{\dim(\mathcal{O}_i)=4} \frac{\langle 0 | \mathcal{O}_i | 0 \rangle}{m_\tau^4} + c \sum_{\dim(\mathcal{O}_i)=6} \frac{\langle 0 | \mathcal{O}_i | 0 \rangle}{m_\tau^6} + \dots, \quad (6.23)$$

where the first term is a kinematical effect proportional to the weighted average ( $\overline{m}^2$ ) of the running masses of the light quarks ( $m_u^2$ ,  $m_d^2$ , and  $m_s^2$ ), and consequently is neglected on the limit of massless quarks. There are also kinematical corrections of order  $1/m_\tau^4$ , and again these are neglected on the limit of massless quarks, so we will ignore them as the most important power corrections arise from the condensates. The quark condensate of dimension  $d=4$  is  $m_q \langle \Psi \overline{\Psi} \rangle$ , which vanishes on the limit of massless quarks. The gluon condensate  $\langle GG \rangle$  of the same dimension is sub-leading in the coupling by  $a^2(m_\tau^2)$  [99], and thus negligible. As a matter of fact, when the  $s$ -dependence of the Wilson coefficients is ignored, the form of the kinematical factor multiplying  $\Pi^{0+1}(s)$  in (6.20) is such that only the operators of dimensions  $d=6$  and  $d=8$  survive the integration [105, 107, 108]. However, when the dependence on  $s$  is considered, other operators contribute, but again they are suppressed by powers of  $a^2(m_\tau^2)$  [99]. The largest power correction turns out to be given by the

$d = 6$  condensate, whereas the  $d = 8$  condensate contributes  $\approx 10^{-5}$ , according to [99, 109].

### 6.3.3 The $R$ -ratio as a contour integral

From the integration of (3.3), one has directly

$$\Pi(s) - \Pi(s') = - \int_{s'}^s \frac{D(-t)}{t} dt, \quad (6.24)$$

where  $s'$  is a reference timelike momentum squared. The  $R$ -ratio  $R(s)$  can be related to the correlator  $\Pi(s)$  using the optical theorem supplemented by analyticity. Thus,

$$R(s) = \frac{1}{\pi} \text{Im} \left\{ \Pi(s+i\epsilon) \right\} = \frac{1}{2\pi i} \left[ \Pi(s+i\epsilon) - \Pi(s-i\epsilon) \right]. \quad (6.25)$$

One can now relate  $R(s)$  to  $D(s)$  using the last two equations,

$$R(s) = - \frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{D(-t)}{t} dt. \quad (6.26)$$

This last expression can be converted into a contour integral, running counterclockwise around the circle  $t = s$ , and avoiding the cut along the positive real axis. Choosing  $-t = se^{i\theta}$ , (6.26) becomes the contour integral

$$R(s) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} D(se^{i\theta}) d\theta. \quad (6.27)$$

### 6.3.4 Calculations in the contour integral representation

As we have seen in the previous sub-sections, a number of observables can be represented as weighted contour integrals, around a circle in the complex  $s$ -plane, of the Adler function  $D(se^{i\theta})$ , which can be generally cast as

$$\hat{R}(s) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} W(\theta) D(se^{i\theta}) d\theta. \quad (6.28)$$

In (6.28) the weight function  $W(\theta)$  specifies the observable,

$$W(\theta) = 1 \quad \longrightarrow \quad \hat{R}(s) = R - \text{ratio } R(s), \quad (6.29)$$

$$W(\theta) = 1 + 2e^{i\theta} - 2e^{i3\theta} - e^{i4\theta} \quad \longrightarrow \quad \hat{R}(s) = D_\tau(s = m_\tau^2). \quad (6.30)$$



If we expand the one loop coupling (1.36)  $a(se^{i\theta})$  in a Taylor expansion in powers of  $\theta$ ,

$$a(se^{i\theta}) = a(s) - i\frac{b}{2}\theta a^2(s) + \left(-i\frac{b}{2}\theta\right)^2 a^3(s) + \left(-i\frac{b}{2}\theta\right)^3 a^4(s) + \dots, \quad (6.31)$$

we see that  $a(se^{i\theta})$  is the effective resummation of terms which arise from the analytical continuation. In fact, if we define

$$\omega_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \theta^n W(\theta) d\theta, \quad (6.32)$$

one has

$$\omega_n = \begin{cases} 0, & n \text{ odd}, \\ \frac{\pi^n}{n+1}, & n \text{ even}, \end{cases} \quad (6.33)$$

for the  $R$ -ratio  $W(\theta)$ , and

$$\omega_1^\tau = \frac{19}{12}i, \quad (6.34)$$

$$\omega_2^\tau = \frac{\pi^2}{3} - \frac{265}{72}, \quad (6.35)$$

$$\omega_3^\tau = \left(\frac{19}{12}\pi^2 - \frac{3355}{288}\right)i, \quad (6.36)$$

$$\omega_4^\tau = \frac{\pi^4}{5} - \frac{265}{36}\pi^2 + \frac{41041}{864}, \quad (6.37)$$

$$\vdots \quad \vdots$$

for the  $R_\tau$   $W(\theta)$ . Thus, we see that the perturbation theory coefficients for both  $R_\tau$  and  $R$ -ratio get corrected by powers of  $b^2\pi^2$  from the analytical continuation. Specifically, for the  $R$ -ratio perturbative coefficients  $r_k$ ,

$$r_1 = d_1, \quad (6.38)$$

$$r_2 = d_2 - \frac{\pi^2}{12}b^2, \quad (6.39)$$

$$\vdots \quad \vdots$$

where the  $d_k$ 's are the perturbative coefficients from the Adler function  $D(a)$ . For the  $R_\tau$  perturbative coefficients  $r_k^\tau$ , one obtains

$$r_1^\tau = d_1 + \frac{19}{24}b, \quad (6.40)$$

$$r_2^\tau = d_2 + \frac{19}{12}d_1b - \left(\frac{\pi^2}{12} - \frac{265}{288}\right)b^2, \quad (6.41)$$

$$\vdots \quad \quad \quad \vdots$$

For the two-loop coupling (1.43), the equivalent of equation (6.31) is slightly more complicated

$$\begin{aligned} a(se^{i\theta}) &= a(s) - i\frac{b}{2}\theta a^2(s) \frac{W_{-1}(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c})}{1+W_{-1}(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c})} \\ &+ \left(-i\frac{b}{2}\theta\right)^2 a^3(s) \frac{W_{-1}^2(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c}) - \frac{1}{2}W_{-1}(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c})}{W_{-1}^2(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c}) + 2W_{-1}(-\frac{1}{e}(s/\tilde{\Lambda}_{\overline{MS}})^{-b/c}) + 1}. \end{aligned} \quad (6.42)$$

Calculating physical observables taking values around the contour in the complex plane requires analytically continuing the Adler function  $D(s)$  for complex values. This problem has been circumvented in the past by calculating Taylor expansions of  $D(se^{i\theta})$  around  $s = s_0$ , and then evaluating the successive derivatives of  $D(s)$  with respect to  $\ln s$  with the help of the RG  $\beta$ -equation [75, 84]. This is no longer necessary because the Lambert W function has well established analytic continuations. The Lambert W function therefore proves all its power and usefulness in the case of contour integral calculations.

The evolution of both the CORGI series  $D(a_0(s))$  in (5.6) and the continued fractions in (5.13) amounts, very simply, to have the coupling  $a_0(s)$  evolving in the complex  $s$ -plane. Since the coupling is given explicitly in (1.43) in terms of the Lambert  $W_{-1}(z)$ , and  $D(a_0(se^{i\theta})) = D(a_0(Qe^{i\theta/2}))$ , with the integration going between 0 and  $\pi$  (the contribution from the lower half-plane is exactly equal) it turns out that we only need the solution of the two-loop RG  $\beta$ -equation in the region where the real part of  $Qe^{i\theta/2}$  is positive. Explicitly,

$$R(s) = \frac{1}{\pi} \int_0^\pi W(\theta) D(a_0(Qe^{i\theta/2})) d\theta, \quad (6.43)$$

where  $a_0(a+bi)$  always involves  $W_{-1}(z)$ . The same thing happens for the Borel sum. However, to use the Borel sum for complex values of the variable one needs to adapt equations (5.7) and (5.8) by replacing

$$Ei\left(-\frac{z_\ell}{x}\right) \longrightarrow -E_1\left(\frac{z_\ell}{x}\right), \quad (6.44)$$

$$Ei\left(\frac{z_\ell}{x}\right) \longrightarrow -E_1\left(-\frac{z_\ell}{x}\right) + i\pi \text{sign}\left[\text{Im}\left(\frac{z_\ell}{x}\right)\right]. \quad (6.45)$$

In order to use the truncated continued function inside the contour integral we start by noting that the  $s$ -dependence is originally contained in  $\rho_0^{(0)} = \frac{b}{2} \ln \frac{s}{\Lambda_{\overline{MS}}^2} - d_1^{\overline{MS}}$ , in such a way that equation (4.18) generalises trivially (see (E.6)), so that *every*  $\rho_0^{(k)C}$  is the real-valued  $\rho_0^{(k)}$  *added* by the complex constant  $ib\theta/2$ . Obviously, since the calculation of the  $\rho_2^{(k)}$ 's does not involve  $\rho_0^{(0)}(s)$ , the former remain real-valued. The only caveat is that one now has to evaluate the sign of the *real part* of  $z$  to decide if the chosen branch of  $G(z)$  for complex values of  $z$  is  $W_0(z)$  or  $W_{-1}(z)$ . Therefore, the generalisation of the truncated continued function method to complex values is also fairly trivial.

## 6.4 Results for the $R_\tau$

In Figure 6.1, the resummation of the  $R_\tau$   $\tilde{R}_\tau(a)$  with all the three resummation methods, compared with the naive sum of the CORGI series, is presented.

The actual values that we get from each resummation method are

$$\tilde{R}_\tau = 0.164 \pm 0.018 \quad (6.46)$$

for the Borel sum (the “error bars” indicated for the Borel sum refer to the ambiguity from the first  $IR$  renormalon),

$$\tilde{R}_\tau = 0.146 \quad (6.47)$$

for the  $P_6^6$  Padé approximant, and

$$\tilde{R}_\tau = 0.152 \quad (6.48)$$

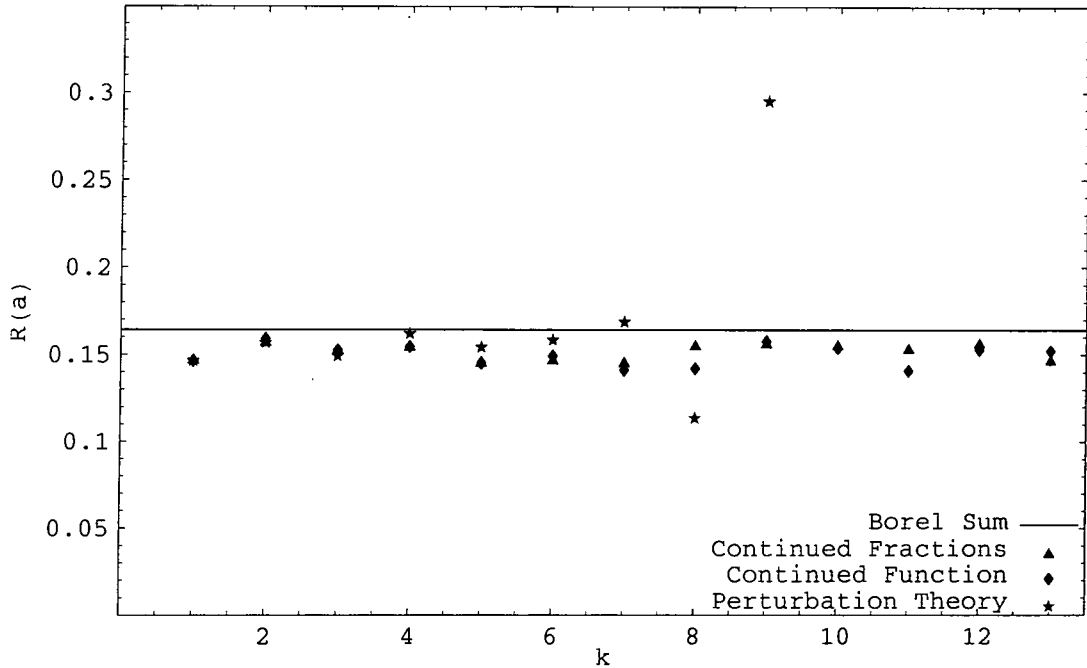


Figure 6.1: The  $R_\tau$   $\tilde{R}_\tau(a)$  in the CORGI scheme resummed with  $Q = m_\tau = 1.777$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 320$  MeV, and  $N_f = 3$  flavours.

for the truncated continued function method at  $\mathcal{O}(a^{14})$ . The values indicated for the Padé approximants and for the truncated continued function method, which are taken at  $\mathcal{O}(a^{14})$ , are not the closest ones to the Borel sum or to the experimental value.

We note that both the Padé approximants and the truncated continued function method give us a sequence of numbers far more stable than the naive sum of the series, which diverges as an alternating-sign factorial.

#### 6.4.1 Results of fitting $\tilde{\Lambda}$ to the $R_\tau$ experimental data

We have chosen values of  $\tilde{\Lambda}$  such that the Borel sum, the  $P_6^6$  Padé approximant, or the truncated continued function method value at  $\mathcal{O}(a^{14})$  coincide with the experimental value  $\tilde{R}_\tau^{exp.}$  or with  $\tilde{R}_\tau^{exp.} + \text{error}$ ,  $\tilde{R}_\tau^{exp.} - \text{error}$ . The results are  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 416_{-30}^{+26}$  MeV for the Borel sum,  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 519 \pm 48$  MeV for the  $P_6^6$  Padé approximant, and  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 516 \pm 48$  MeV for the truncated continued function method at  $\mathcal{O}(a^{14})$ . For

the NNLO perturbation theory result we obtain  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 475_{-45}^{+44}$  MeV.

The results mentioned in the previous paragraph allow us to calculate the following values for the coupling constant at the EC scale:  $a_0(m_\tau) = 0.126 \pm 0.006$  from the Borel sum  $\tilde{\Lambda}_{\overline{MS}}^{(3)}$ ,  $a_0(m_\tau) = 0.150_{-0.011}^{+0.012}$  from the Padé approximant  $P_6^6 \tilde{\Lambda}_{\overline{MS}}^{(3)}$ , and  $a_0(m_\tau) = 0.149_{-0.011}^{+0.012}$  from the truncated continued function method  $\tilde{\Lambda}_{\overline{MS}}^{(3)}$  at  $\mathcal{O}(a^{14})$ . For the NNLO perturbation theory result we obtain  $a_0(m_\tau) = 0.139 \pm 0.010$ .

For the more widely quoted physical scale  $\alpha_s(\mu = Q = m_\tau)$ , we obtain:  $\alpha_s(m_\tau) = 0.360_{-0.020}^{+0.021}$  for the truncated continued function method  $\tilde{\Lambda}_{\overline{MS}}^{(3)}$  at  $\mathcal{O}(a^{14})$ ,  $\alpha_s(m_\tau) = 0.361_{-0.020}^{+0.021}$  for the  $P_6^6$  Padé approximant  $\tilde{\Lambda}_{\overline{MS}}^{(3)}$ , and finally  $\alpha_s(m_\tau) = 0.318_{-0.012}^{+0.011}$  for the Borel sum  $\tilde{\Lambda}_{\overline{MS}}^{(3)}$ . For the NNLO perturbation theory result we obtain  $\alpha_s(m_\tau) = 0.343 \pm 0.019$ .

The results for  $\alpha_s(m_\tau)$  available in the literature quote  $\alpha_s(m_\tau) = 0.351 \pm 0.008$  with experimental error alone, and  $\alpha_s(m_\tau) = 0.35 \pm 0.03$  with the theoretical error estimated from 40% of the  $\alpha_s^3(m_\tau)$  contribution and the nonperturbative contributions [83]. Other determination taking into account renormalon contributions yields  $\alpha_s(m_\tau) = 0.303$  [104] (on the V-scheme). Another approach based on the EC  $\beta$ -function yields  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 429 \pm 12$  MeV and  $\alpha_s(m_\tau) = 0.339 \pm 0.006$  with the Borel sum, and  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 450 \pm 16$  MeV yielding  $\alpha_s(m_\tau) = 0.350 \pm 0.008$  for the NNLO [75].

## 6.5 Results for the R-ratio decay rate

In Figure 6.2 we display the performance of our three resummation methods for the R-ratio  $R(a)$  at a high energy. The naive sum of the perturbation theory diverges in the familiar fashion, but, reflecting the smallness of the coupling, the divergence only starts to show at  $\mathcal{O}(a^{28})$ .

In Figure 6.3 we can see the very same at an intermediate energy. It must be noted that the Padé approximants converge to the Borel sum, whereas the truncated continued function method chooses a slightly different limit, and remains stable to three digits from  $\mathcal{O}(a^{22})$ .

In Figure 6.4 we show the result of doing the same exercise with the low energy conditions. Opposite what happens with the naive perturbation theory series (which diverges rapidly), both the Padé approximants and the truncated continued function

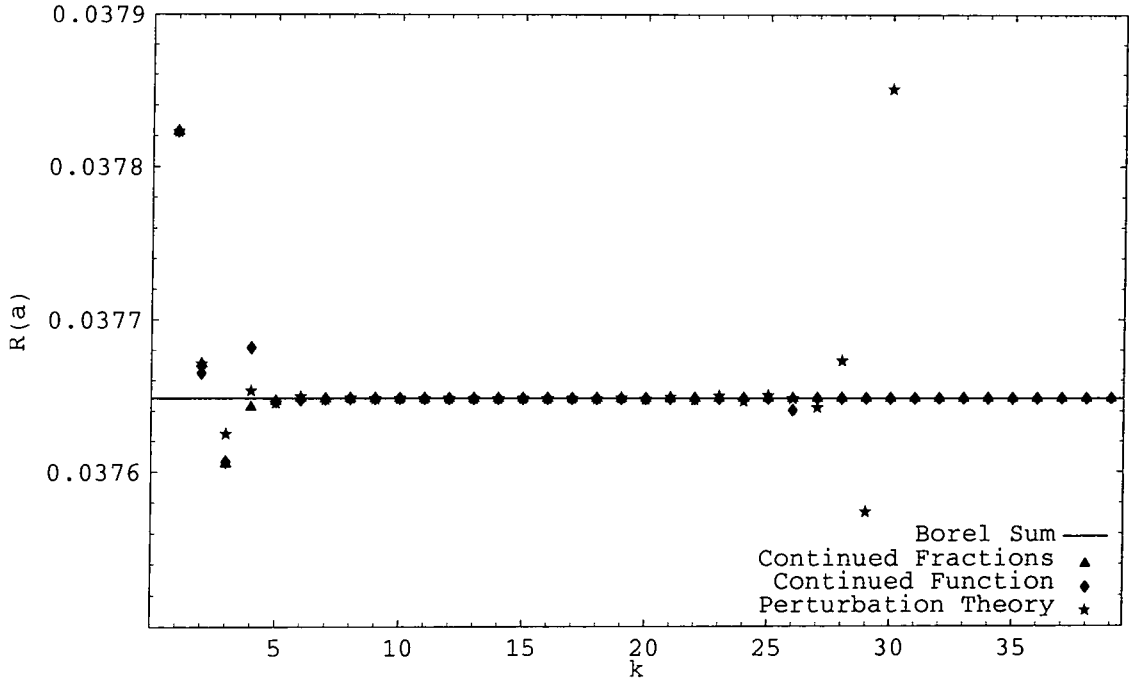


Figure 6.2: The  $R$ -Ratio  $R(a)$  in the CORGI scheme resummed with  $Q = 91$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 200$  MeV, and  $N_f = 5$  flavours.

method remain within a bounded region. However, unlike the Borel sum, the two order-by-order methods do not provide us with a single value to more than a digit. One can say that the truncated continued function method suggests  $R \simeq 0.9 \pm 0.05$ , but this is more a statistics statement than a limiting value.

### 6.5.1 Results of fitting $\tilde{\Lambda}$ to the R-ratio experimental data

At the energy of the  $Z^0$  (91.2 GeV), all the resummation methods converge to an indistinguishable limit, which the four digits precision being reached at such low orders as  $\mathcal{O}(a^4)$  for the Padé approximants, and  $\mathcal{O}(a^6)$  for the truncated continued function method. The NNLO calculation ( $= a_0 + X_2 a_0^3$ ) differs from this limit by a mere  $\delta = 2.29 \times 10^{-5}$ . So, at this very high energy, there is no difference worth mentioning.

Fitting  $\tilde{\Lambda}_{\overline{MS}}^{(5)}$  so that any of the resummation methods coincides with the experimental value (6.13) gives

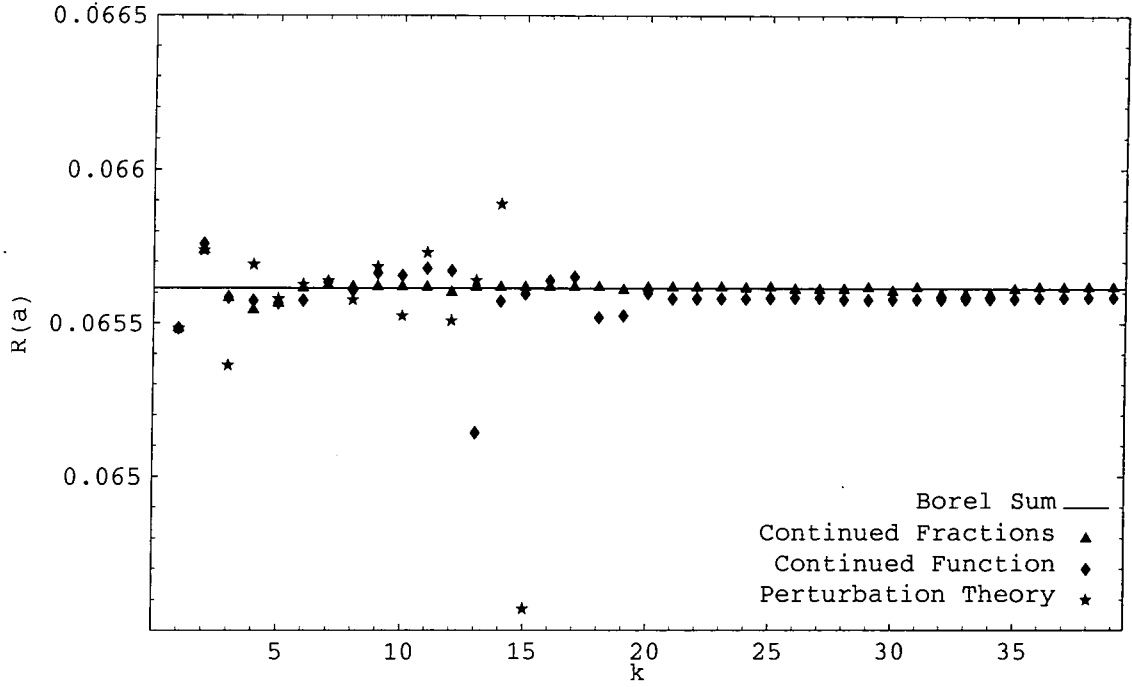


Figure 6.3: The  $R$ -Ratio  $R(a)$  in the CORGI scheme resummed with  $Q = 5$  GeV,  $\tilde{\Lambda}_{\overline{MS}} = 279$  MeV, and  $N_f = 4$  flavours.

$$\tilde{\Lambda}_{\overline{MS}}^{(5)} = 299_{-7}^{+6} \text{ MeV}. \quad (6.49)$$

With this value of  $\tilde{\Lambda}_{\overline{MS}}^{(5)}$  in hand, one calculates the two-loop EC scale coupling as  $a_0(m_{Z^0}) = 0.0411 \pm 0.0001$ . The two-loop physical scale coupling yields  $\alpha_s(m_{Z^0}) = 0.1218 \pm 0.0004$ . This is compatible with the value quoted in the literature,  $\alpha_s(m_{Z^0}) = 0.124 \pm 0.005$  [110] for this physical process.

## 6.6 Evolution of the coupling between the $\tau$ and $Z_0$ energies

Given the calculations of the coupling  $\alpha_s(m_\tau)$  from the  $R_\tau$  and from the  $R$ -ratio at the  $Z_0$  pole, one would naturally wish to compare the two, and preferably at the  $m_{Z^0}$  energy at which the value of  $\alpha_s$  is usually cited in the literature of the field. We will follow [111, 112] in evolving the coupling between the  $\tau$  and  $Z_0$  energies. Their

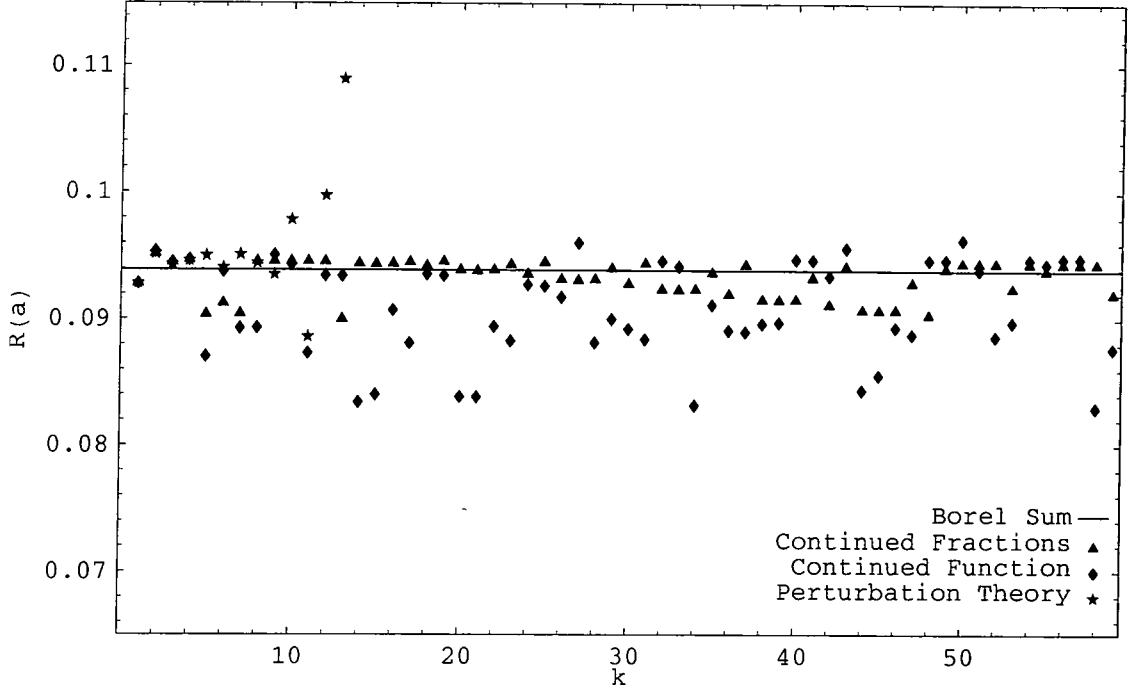


Figure 6.4: The  $R$ -Ratio  $R(a)$  in the CORGI scheme resummed with  $Q=1.5$  GeV,  $\tilde{\Lambda}_{\overline{MS}}=320$  MeV, and  $N_f=3$  flavours.

solution for the four loops RG equation [112] is

$$a(\mu) = a^{(2)}(\mu) \left( 1 + c_2(\mu)[a^{(2)}(\mu)]^2 + c_3(\mu)[a^{(2)}(\mu)]^3 \right), \quad (6.50)$$

where  $a^{(2)}(\mu)$  is the approximate two-loop solution

$$a^{(2)}(\mu) = \frac{a(\mu_0)}{K + ca(\mu_0)L + c^2a^2(\mu_0)(1-K+L)/K}, \quad (6.51)$$

and

$$c_2(\mu) = c_2^{\overline{MS}}(1-K), \quad (6.52)$$

$$c_3(\mu) = \frac{c_3^{\overline{MS}}}{2}(1-K^2) + cc_2^{\overline{MS}}K(K-1-L) + \frac{c^3}{2}(L^2 - (1-K)^2), \quad (6.53)$$

with  $K = 1 + ba(\mu_0)\ln \frac{\mu}{\mu_0}$ ,  $L = \ln K$ .

It must be noted that we will take  $a^{2-loops}(\mu_0 = m_\tau)$  (thus at the physical scale) as the initial condition. The transition through the quark thresholds  $\mu_{th}$  (for the charm



and bottom quarks) is made with the matching condition [112]

$$a_{N_f}(\mu_{th}) = a_{N_f-1}(\mu_{th}) \left[ 1 + \sum_{k=1}^{+\infty} C_k(x) a_{N_f-1}^k(\mu_{th}) \right], \quad (6.54)$$

where  $x = \ln(\mu_{th}/m)$ , and  $m$  is some RG-invariant mass of the heavy quark (the charm or the bottom in our case).

We obtain, as a result of evolving  $\alpha_s(m_\tau)$  to the  $m_{Z^0}$  energy, the following results. For the truncated continued function method at  $\mathcal{O}(a^{14})$ , the result is indistinguishable from the value for the  $P_6^6$  Padé approximant,

$$\alpha_s(m_{Z^0}) = 0.123 \pm 0.002. \quad (6.55)$$

The result for the Borel sum is

$$\alpha_s(m_{Z^0}) = 0.119 \pm 0.001. \quad (6.56)$$

Finally, the result for the NNLO is

$$\alpha_s(m_{Z^0}) = 0.122 \pm 0.002. \quad (6.57)$$

Again, the value quoted in the literature is [110]

$$\alpha_s(m_{Z^0}) = 0.124 \pm 0.005, \quad (6.58)$$

which is very close to our truncated continued function method or the Padé approximant result, which are slightly better than the Borel sum value. The difference  $G(\mathcal{O}(a^{14})) - NNLO \equiv \delta$  is too small ( $\delta = 1.82 \times 10^{-4}$ ) to allow us any definite conclusions about the importance of the renormalons contributions at this energy.

The effect of varying the masses of the charm and bottom quarks on the physical ranges 1.15 – 1.35 GeV and 4.0 – 4.4 GeV (respectively) amounts to variations of less than 1% in the final result. The parameters  $\mu_{th}$  also have a negligible influence in the final result, as long as they are kept close to the masses of the heavy quarks at the respective thresholds ( $(\mu_{th}/m) \approx 1$ ).

# Chapter 7

## Discussion and conclusions

Having reviewed in the opening Chapter the rudiments of perturbative QCD, we saw in Chapters 2 and 3 that the resulting standard perturbative series in the renormalised coupling is unfortunately afflicted with factorial growth of the coefficients and is an asymptotic series. As we discussed and reviewed, one can understand this large-order behaviour in terms of ultra-violet and infra-red renormalons, although a complete diagrammatic understanding is lacking in the QCD case. Whilst, as we reviewed in Chapter 2, one can use Borel summation to define a sum of the series, and improve convergence by using Padé approximation rather than straightforward truncation of the perturbation series, these are *ad hoc* mathematical techniques applied to tame an asymptotic series. In this Thesis we instead focused on a physically-motivated reformulation in which perturbation theory is formulated not in terms of the renormalised coupling, but instead in terms of the dimensional transmutation parameter (e.g.  $\tilde{\Lambda}_{\overline{MS}}$ ) of the theory, and the fundamental property of universal asymptotic scaling (equation (4.9)) which provides an operational definition of it. The violation of scaling at finite energy is controlled by an effective charge which also satisfies the asymptotic scaling property with violations which can be described by yet another effective charge, and so on. This self-similar construction naturally gives rise to a continued function representation of QCD observables, the iterated function being that involved in the solution of the two-loop  $\beta$ -function equation. In the leading- $b$  limit (for a one loop  $\beta$ -function) the continued function reduces to a continued fraction, and the successive “natural sequence” approximants are simply diagonal Padé approximants of the original perturbation series. The main aim of the

This thesis was to study the convergence properties of the successive approximants in the continued function approach. We use the leading- $b$  approximation, whose motivation is discussed in section 3.2, to model the higher-order corrections. The CORGI formalism is used to avoid RS-dependence in the resummation when the exact NLO and NNLO corrections are included. This makes the RS-invariant all-orders resummation much easier to perform than the analogous previous investigations using the effective charge formalism [75, 84].

Numerical studies were performed in Chapter 5 for the Adler D-function of QCD vacuum polarisation and for the polarised Bjorken and Gross-Llewellyn Smith sum rules. Fixed-order CORGI perturbation theory, diagonal and off-diagonal Padé approximants for the CORGI perturbation series and the continued function approximants are plotted and compared with the all-orders Borel sum of the CORGI series, regulated with the Cauchy Principal Value to control the infra-red renormalons. The studies are performed at various energies. Fixed-order perturbation theory necessarily breaks down at some point, reflecting the dominant ultra-violet or infra-red renormalon singularities. At higher energies for the Adler function both the Padé approximants and the continued function approximants appear to converge very nicely to the same value as the Borel sum. If the series is a Stieltjes one can prove convergence of the diagonal Padé approximants to the same limit as the Borel sum, as discussed in Chapter 2. However, infra-red renormalons are also present. Singularities on the positive axis in the Borel plane translate into closely-spaced real positive zeros in the denominators of the diagonal Padé approximants, and the convergence is destabilised. This feature is very evident in the figures, particularly for the deep inelastic sum rules where the asymptotics are controlled by a leading infra-red renormalon, and for the Adler D function at low energies. In contrast, however, the continued function approximants in suitably high orders remain stable, typically at the two or three significant figure level. They approach a stable value *different* from the Cauchy Principal Value regulated Borel sum. This difference is of the same order as the leading power correction associated with the observable, which in turn will depend on the position of the leading infra-red renormalon. It seems that the continued function approach builds power corrections from the perturbative information. We did not attempt to analyze the way in which this intriguing effect

comes about. That this might not be impossible is suggested by the following. The function  $G(z)$  which is iterated is closely related to the Lambert  $W$  function. As discussed in [16], the resulting  $Q$ -dependence of  $G$  gives a causal analyticity structure. In our approach the causal structure of the observable  $R(Q)$  naturally results from iterating that of  $G$ , although the details would be very complicated to work through. Unfortunately we were unable to derive any convergence proofs for the procedure, although in Chapter 4 we gave an outline of how such convergence might operate.

The results of the fittings for contour-improved observables based on the Adler function  $D(a)$  were seen to be sensible for all the resummation methods used. Fitting the continued function approximants to the experimental results from the  $\tau$  lepton decay, the dimensional transmutation parameter was found to be  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 516 \pm 48$  MeV. This amounts to a value of  $\alpha_s(\mu=m_\tau) = 0.360_{-0.020}^{+0.021}$  for the coupling. Another approach also based on RS-invariants, but in the EC scheme and which uses the Borel sum [75] gives  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 429 \pm 12$  MeV yielding  $\alpha_s(m_\tau) = 0.339 \pm 0.006$ . Albeit this is slightly different from our results with the truncated continued function method, it is very close to our results with the Borel sum, which are  $\tilde{\Lambda}_{\overline{MS}}^{(3)} = 416_{-30}^{+26}$  MeV and  $\alpha_s(m_\tau) = 0.318_{-0.012}^{+0.011}$ . Another reference [104] also based on the Borel sum but which uses the V-scheme and no RS-invariants gives  $\alpha_s(m_\tau) = 0.303$ . We estimate the contribution from renormalons to the coupling  $\alpha_s$  (obtained by comparison with the NNLO result), as  $\delta \simeq 10^{-2}$  at the  $\tau$  energy. We note that our result with the truncated continued function method is the closest to the value quoted in the literature,  $\alpha_s(m_\tau) = 0.351 \pm 0.008$  [83].

For the  $R$ -ratio at the  $m_{Z^0}$  energy, the agreement is much better. Our result is  $\tilde{\Lambda}_{\overline{MS}}^{(5)} = 299_{-7}^{+6}$  MeV, yielding  $\alpha_s(m_{Z^0}) = 0.1218 \pm 0.0004$ . Unsurprisingly, giving such a high energy, this is very close to results quoted in the literature such as  $\alpha_s(m_{Z^0}) = 0.124 \pm 0.005$  [110], and  $\alpha_s(m_{Z^0}) = 0.123 \pm 0.004$  [83] for this physical process. The evolution of our truncated continued function method result from the  $m_\tau$  to the  $m_{Z^0}$  energy is virtually indistinguishable  $\alpha_s(m_{Z^0}) = 0.123 \pm 0.002$ . All of these results fall within the so-called “world average”  $\alpha_s(m_{Z^0}) = 0.1192 \pm 0.0028 \pm 0.002(\text{scale})$  [83].

# Appendix A

## Special functions

In this Appendix we shall provide the definitions and approximations [113] of some of the special functions which are used in this Thesis.

### The $\Gamma$ function

The function  $\Gamma(z)$  is defined as

$$\Gamma(z) \equiv \int_0^{+\infty} t^{z-1} e^{-t} dt \quad (Re(z) > 0), \quad (\text{A.1})$$

and it is asymptotically approximated at large  $z$  by

$$\Gamma(az + b) \simeq \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \quad (|arg(z)| < \pi, a > 0). \quad (\text{A.2})$$

The following expansion stands

$$\Gamma(1 + \epsilon) = 1 - \gamma_E \epsilon + \frac{1}{2} \left( \gamma_E^2 + \frac{\pi^2}{6} \right) \epsilon^2 + \mathcal{O}(\epsilon^3), \quad (\text{A.3})$$

where  $\gamma_E = 0.577216 \dots$  is the Euler-Mascheroni constant. The formula

$$\Gamma(\epsilon)(4\pi)^\epsilon = \frac{1}{\epsilon} + \ln 4\pi - \gamma_E + \mathcal{O}(\epsilon), \quad (\text{A.4})$$

is often needed in applications. For integer values, the  $\Gamma$  function is simply related to the factorial,

$$\Gamma(k+1) = k!, \quad (\text{A.5})$$

and thus, for large  $k$  (A.2) becomes the Stirling approximation

$$k! \simeq \sqrt{2\pi k} \left(\frac{k}{e}\right)^k. \quad (\text{A.6})$$

The integral definition of the factorial is obviously

$$k! \equiv \int_0^{+\infty} t^k e^{-t} dt. \quad (\text{A.7})$$

## The Riemann $\zeta$ function

The Riemann  $\zeta$  function is defined as:

$$\zeta_\ell \equiv \sum_{n=1}^{+\infty} n^{-\ell}, \quad (\text{A.8})$$

and  $\zeta_3 = 1.202057 \dots$ ,  $\zeta_5 = 1.036928$ ,  $\zeta_7 = 1.008349 \dots$

## The exponential integral functions

The exponential integral function  $Ei(x)$  is defined as:

$$Ei(x) \equiv - \int_{-x}^{+\infty} \frac{e^{-t}}{t} dt \quad (x < 0). \quad (\text{A.9})$$

The exponential integral function can be extended for positive argument ( $x > 0$ ) by arbitrarily choosing the Cauchy Principal Value as the prescription.

The generalised exponential integral function  $E_n(x)$  is defined as

$$E_n(x) \equiv \int_1^{+\infty} \frac{e^{-xt}}{t^n} dt \quad (Re(x) > 0; n \in \mathcal{N}_0), \quad (\text{A.10})$$

and one has the trivial relation  $Ei(x) = -E_1(-x)$  for real and negative  $x$  ( $Ei(x) \pm i\pi = -E_1(-x \pm i\epsilon)$  for real and positive  $x$ ):

The analytical continuation of the generalised exponential integral function is

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} \left( -\ln z - \gamma_E + \sum_{m=1}^{n-1} \frac{1}{m} \right) - \sum_{\substack{m=0 \\ m \neq n-1}}^{+\infty} \frac{(-z)^m}{(m-n+1)m!} \quad (|arg(z)| < \pi). \quad (\text{A.11})$$

## The dilogarithm function

The dilogarithm function  $L_2(x)$  is defined as:

$$L_2(x) \equiv - \int_0^x \frac{\ln(1-y)}{y} dy. \quad (\text{A.12})$$

## The Elliptic Theta function $\vartheta_3(u, q)$

The Elliptic Theta function  $\vartheta_3(u, q)$  is defined [114] as:

$$\vartheta_3(u, q) \equiv 1 + 2 \sum_{n=1}^{+\infty} q^{n^2} \cos(2nu). \quad (\text{A.13})$$

# Appendix B

## The Lambert W function

The function to which this Appendix is devoted is related to studies done as early as the third quarter of the eighteenth century by Euler and Lambert. The notation and standardisation of this function used here are recent [115], and the relevance to particle physics of the Lambert W function was not recognized until even more recently [16].

**Definition 7 : Lambert W function** *The Lambert W function, denoted  $W(z)$ , is defined as the many-valued roots of*

$$W(z)e^{W(z)} = z. \quad (\text{B.1})$$

The Lambert W function has a numerable infinity of branches  $W_k(z)$  (for  $k \in \mathbb{Z}$ ), of which only two take real values in some part of their domains:  $W_0(z)$ , for  $z \in [-\frac{1}{e}, +\infty[$ ; and  $W_{-1}(z)$ , for  $z \in [-\frac{1}{e}, 0]$ . These two branches are jointly plotted in Figure B.1.

The following expansion of  $W_0(z)$  around  $z=0$  is used in this thesis:

$$W_0(z) = z - z^2 + \frac{3}{2}z^3 - \frac{8}{3}z^4 + \frac{125}{24}z^5 - \frac{54}{5}z^6 + \frac{16807}{720}z^7 + \mathcal{O}(z^8). \quad (\text{B.2})$$

The expansion of  $W_{-1}(z)$  around the branch-point ( $z = -\frac{1}{e}$ ) is of special interest. We quote here the expansion of  $W_{-1}(-e^{-z-1})$  around  $z=0$ , because with this form of the argument the expansion is particularly simple:



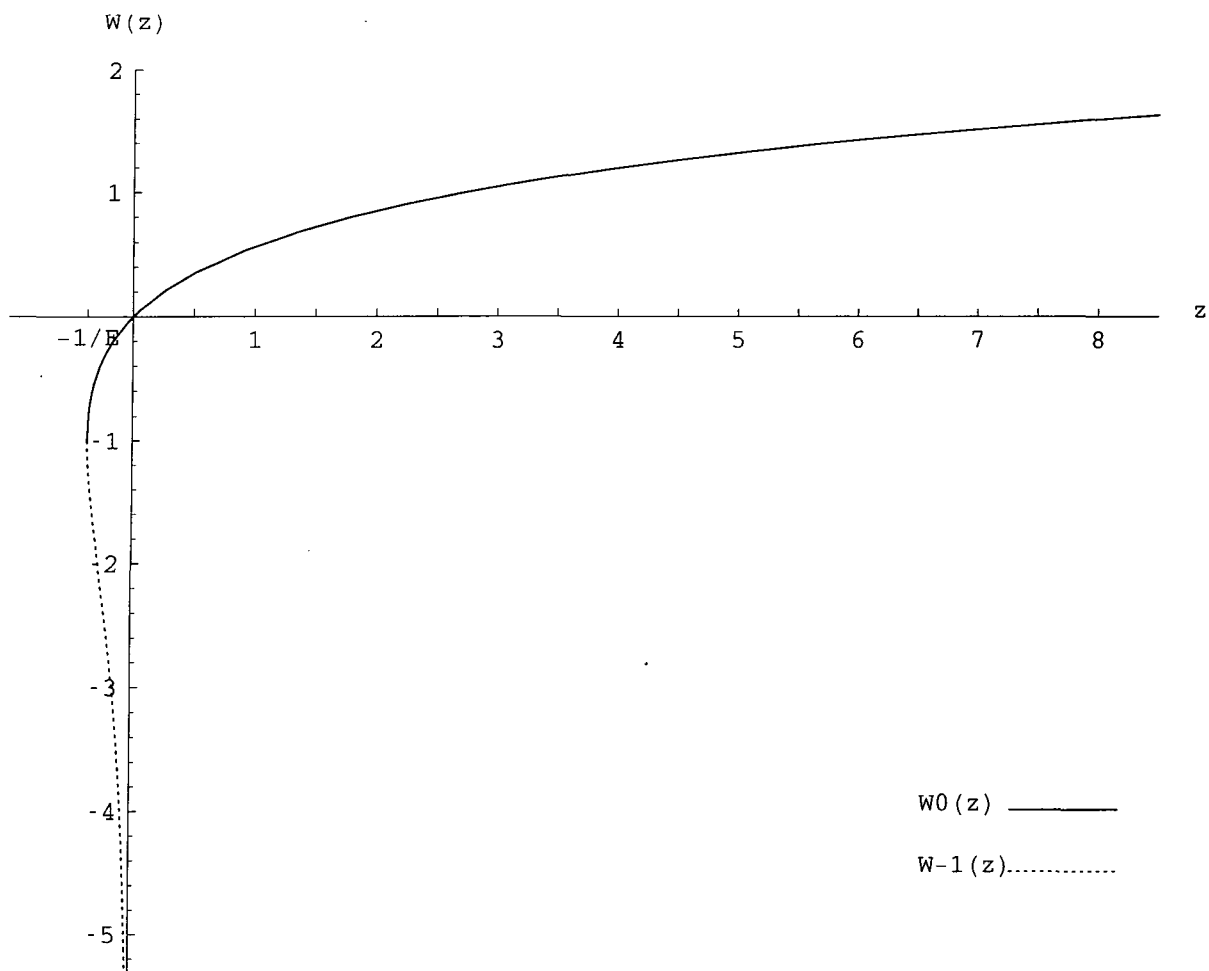


Figure B.1: The real-valued branches of the Lambert W function.

$$W_{-1}(-e^{-z-1}) = -1 + \sqrt{2z} - \frac{2z}{3} + \frac{z^{3/2}}{9\sqrt{2}} + \frac{2z^2}{135} + \frac{z^{5/2}}{540\sqrt{2}} - \frac{4z^3}{8505} + \mathcal{O}(z^{7/2}). \quad (\text{B.3})$$

The Lambert W function is implemented in both Mathematica 3.0 and Maple V. The respective commands are:  $W_k(z) = \text{ProductLog}[k, z]$ ,  $W_0(z) = \text{ProductLog}[z]$ ; and  $W_k(z) = \text{LambertW}(k, z)$ ,  $W_0(z) = \text{LambertW}(z)$ . The analytical continuations are also implemented and satisfy the property  $W_k(z) = W_{-k}^*(z^*)$  for all  $k$  other than 0 and 1. However, one has  $W_1(z) = W_{-1}^*(z^*)$  everywhere except for a cut on the negative real semi-axis.

For other properties of the Lambert W function, we refer the reader to [115]. Some applications in several areas of scientific knowledge were discussed in [116].

# Appendix C

## Reversion of a power series

Given a generic power series

$$R(a) = d_0 a + d_1 a^2 + \dots d_k a^{k+1} + \dots, \quad (\text{C.1})$$

its reversed power series  $a(R)$  can be defined as

$$a(R) = K_1 R + K_2 R^2 + \dots K_k R^k + \dots \quad (\text{C.2})$$

To find the relations between the coefficients  $K_k$  and the coefficients  $d_k$  of the original series, one can simply replace (C.1) in (C.2) and equate powers of  $a$ . This brute force method becomes cumbersome at  $\mathcal{O}(a^{16})$  using Mathematica. Fortunately, an alternative method exists, giving the coefficients  $K_k$  of the reversed series as

$$K_k = \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \frac{t}{D(t)} \Big|_{t=0}, \quad (\text{C.3})$$

(this formula can be obtained with the residues theorem). This second method becomes cumbersome at  $\mathcal{O}(a^{20})$  with the same software. With any of this methods, one has:

$$\begin{aligned} K_1 &= d_0^{-1}, \\ K_2 &= -d_0^{-3} d_1, \\ K_3 &= d_0^{-4} (2d_0^{-1} d_1^2 - d_2), \\ K_4 &= d_0^{-5} (5d_0^{-1} d_1 d_2 - 5d_0^{-2} d_1^3 - d_3), \end{aligned}$$

$$\begin{aligned}
K_5 &= d_0^{-6}(14d_0^{-3}d_1^4-21d_0^{-2}d_1^2d_2+3d_0^{-1}d_2^2+6d_0^{-1}d_1d_3-d_4) \\
&\vdots \qquad \vdots
\end{aligned}
\tag{C.4}$$

# Appendix D

## Renormalonic integrals

Let us start by considering first the integrals in the infrared region. By integration by parts, it is easy to prove that:

$$\int_0^{\frac{\mu^2}{Q^2}} x^\ell \ln^k \left( x \frac{Q^2}{\mu^2} \right) dx = - \left( -\frac{1}{\ell+1} \right)^{k+1} k! \left( \frac{\mu^2}{Q^2} \right)^{\ell+1} \quad (\ell \geq 0), \quad (\text{D.1})$$

$$\int_0^{\frac{\mu^2}{Q^2}} x^\ell \ln x \ln^k \left( x \frac{Q^2}{\mu^2} \right) dx = - \left( -\frac{1}{\ell+1} \right)^{k+2} \left( k+1 - (\ell+1) \ln \frac{\mu^2}{Q^2} \right) k! \left( \frac{\mu^2}{Q^2} \right)^{\ell+1} \quad (\ell \geq 0). \quad (\text{D.2})$$

We are also interested in integrals in the ultraviolet region. Also with integration by parts, it is found that

$$\int_{\frac{\mu^2}{Q^2}}^{\Lambda^2} \frac{\ln^k \left( x \frac{Q^2}{\mu^2} \right)}{x^\ell} dx = \left( \frac{1}{\ell-1} \right)^{k+1} k! \left( \frac{Q^2}{\mu^2} \right)^{\ell-1} \quad (\ell \geq 2), \quad (\text{D.3})$$

$$\int_{\frac{\mu^2}{Q^2}}^{\Lambda^2} \ln x \frac{\ln^k \left( x \frac{Q^2}{\mu^2} \right)}{x^\ell} dx = \left( \frac{1}{\ell-1} \right)^{k+2} \left( k+1 + (\ell-1) \ln \frac{\mu^2}{Q^2} \right) k! \left( \frac{Q^2}{\mu^2} \right)^{\ell-1} \quad (\ell \geq 2), \quad (\text{D.4})$$

where we took the upper limit  $\Lambda^2 \rightarrow +\infty$  upon integration.

# Appendix E

## An alternative algorithm to calculate $\rho_0^{(k)}$ 's and $\rho_2^{(k)}$ 's

Let us consider a perturbation theory expansion with complete renormalisation group improvement (CORGI) performed:

$$R^{(0)}(a_0) = a_0 + X_2^{(0)} a_0^3 + X_3^{(0)} a_0^4 + \dots X_k^{(0)} a_0^{k+1} + \dots \quad (\text{E.1})$$

Since we have resummed all the  $d_1$ 's, the coupling here is  $a_0$ , from (1.91).

The initial  $\rho_0^{(0)} (= b \ln \frac{Q}{\Lambda_{QCD}} - d_1 = F(a_0))$  and  $\rho_2^{(0)} (= X_2^{(0)} = d_2 - c d_1 - d_1^2 + c_2)$  are known for most observables.  $X_3^{(0)}, X_4^{(0)}, \dots$ , in general, will only be known in the leading- $b$  approximation. Now, inverting each equation in the cascade of equations in (4.19), we have

$$R^{(k)} = \frac{\rho_0^{(k-1)} - F(R^{(k-1)}(a_0))}{\rho_2^{(k-1)}}, \quad (\text{E.2})$$

at each step, but  $\rho_0^{(k-1)} = \rho_0^{(0)} - X_1^{(k-1)}$  (for  $k \geq 1$ , since  $X_1^{(0)} \equiv 0$ ), so we obtain an expression which is expandable as a power series in  $a_0$ :

$$R^{(k)}(a_0) = \frac{F(a_0) - X_1^{(k-1)} - F(a_0 + X_1^{(k-1)} a_0^2 + \dots)}{\rho_2^{(k-1)}} \equiv a_0 + X_1^{(k)} a_0^2 + X_2^{(k)} a_0^3 + \dots, \quad (\text{E.3})$$

where the right-hand side must be taken as the definition of the new  $X_1^{(k)}$  and  $X_2^{(k)}$ , which are calculated at each step. From these, we can obtain

$$\rho_0^{(k)} = \rho_0^{(k-1)} - X_1^{(k)}, \quad (\text{E.4})$$

$$\rho_2^{(k)} = X_2^{(k)} - cX_1^{(k)} - X_1^{(k)^2}. \quad (\text{E.5})$$

Then, we can replace  $\rho_2^{(k)}$  and  $X_1^{(k)}$  in (E.3) to calculate  $\rho_0^{(k+1)}$  and  $\rho_2^{(k+1)}$ , and the process could go on indefinitely.

The extension to complex values,  $\rho_0^{(k)\mathcal{C}}$  presents no difficulties, as the net effect of  $Q^2 \rightarrow Q^2 e^{i\theta}$  is to add an imaginary constant to any real-valued  $\rho_0^{(k)}$ :

$$\rho_0^{(k)} \rightarrow \rho_0^{(k)} + \frac{ib\theta}{2} \equiv \rho_0^{(k)\mathcal{C}}. \quad (\text{E.6})$$

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